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# Comtrans algebras

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# Comtrans algebras

by

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## GENERAL INTRODUCTION

This dissertation consists of two major parts. In the first part we study the comtrans algebras, which were introduced in [2], from a purely algebraic point of view, working toward the classifcation of simple comtrans algebras over suitable fields. Comtrans algebras are defined in the second section, and three broad classes of comtrans algebras are represented by the examples of that section. In the third section, simple comtrans algebras are defined, and it is shown that the simplicity of a Lie algebra and the simplicity of the comtrans algebras which is derived from the Lie algebra are equivelent. In the fourth section, a necessary and sufficient condition for the comtrans algebra $CT(A, B)$ of example 2.1 to be simple is provided. In the final section, it is shown that the series of simple algebras $CT(A, B)$ and $CT(L)$ of example 2.2 are distinct, with the exception of the comtrans algebra given by the vector triple product.

In Part II, the framework for the representation theory of comtrans algebras is established. In the second section, centrality theory from the universal algebraic field of Mal'cev varieties (cf. [3]) is used to get an elementary description of $E$-modules. In the third section it is shown that the representation theory of a comtrans algebra $E$ is equivalent to the representation theory of an associative universal envoloping algebra $\widetilde{M}_E$ of $E$. In the final section, the universal enveloping algebra of a comtrans algebra over a field is identified as the tensor algebra over $(E \wedge E) \oplus (E \otimes E) \oplus (E \otimes E)$.

In Part III, a comtrans algebra $CT(E, \beta)$ form a pair $(E, \beta)$ consisting of a unital module $E$ over a commutative ring $R$ with 1 and a bilinear form $\beta \colon E^2 \to R$ is produced. A "transposed" comtrans algebra $CT(E, \beta)^\tau$ is also given by the pair

$(E,\beta)$. In the second section, some elementary topics such as the notaions of ideal, abelian algebras, and simple algbras are recalled. The transposition relationship between a pair of comtrans algebras, typified by $CT(E,\beta)$ and $CT(E,\beta)^{\tau}$, is also described. In the third section, the basic construction of the comtrans algebras $CT(E,\beta)$ and $CT(E,\beta)^{\tau}$ from a module $(E,\beta)$ with a bilinear form is given. It is shown how simplicity of the comtrans algebras is equivalent to non-degeneracy of the form and simplicity of the ring of scalars. It is also shown that the automorphism groups of the formed space $(E,\beta)$ and of the comtrans algebras $CT(E,\beta)$ and $CT(E,\beta)^{\tau}$ coincide. The final section is concerned with the problem of recognizing when a comtrans algebra is a "form algebra". The answer is given in Theorem 4.1.

## Explanation of Dissertation Format

The dissertation is written in the alternate dissertation format. Each part represents a paper which has been submitted to a scholarly journal for publication. A general summary is included at the end of this dissertation. Each paper includes an individual bibliography.

# PART I. SIMPLE COMTRANS ALGEBRAS


## Abstract

Simple comtrans algebras determined by Lie algebras and by pairs of matrices are characterized. The two classes separate, except for the vector triple product algebra.


## 1. Introduction

Comtrans algebras were introduced in [5] as part of an answer to a problem in differential geometry [1, Problem X.3.9] [4, p.16]: finding the algebraic structure in the tangent bundle that locally determines the coordinate $n$-ary loop of an $(n+1)$-web [2,§3.7]. Loosely speaking, the relationship of comtrans algebras to smooth $n$-loops is analogous to the relationship of Lie algebras to Lie groups. The purpose of the current paper is to begin a study of comtrans algebras from a purely algebraic point of view, working towards the classification of simple comtrans algebras over suitable fields.

Comtrans algebras are defined in the second section. Three broad classes of comtrans algebras are represented by the examples of that section: the comtrans algebra $CT(A, B)$ of Example 2.1 furnished by a pair $A, B$ of square matrices, the comtrans algebras $CT(L)$ of Example 2.2 given by repeated commutators in a Lie algebra L, and the comtrans algebras $CT(E, f)$ of Example 2.3 given by a symmetric bilinear form $f : E \times E \to R$. Example 2.3 is of independent interest as showing how comtrans algebras furnish an intrinsic algebraisation of a symmetric bilinear form, producing the algebra structure directly on the space of definition $E$ of the form. (By contrast, other algebraisations of symmetric bilinear forms, such as those given by Jordan algebras or Clifford algebras, require an extension of the underlying space.) Nevertheless, this aspect of the $CT(E,f)$ will not be pursued at present. Their role here is purely auxiliary to the main topic, which is the identification of the simple algebras $CT(A,B)$ and $CT(L)$. Simple comtrans algebras are defined in Section 3, which then shows (Theorem 3.2) that the simplicity of a Lie algebra L and the simplicity of the corresponding comtrans algebra $CT(L)$ are equivalent. The fourth section is devoted to the proof of the characterization Theorem 4.7 for simple comtrans algebras $CT(A,B)$. The final section shows that the series of simple algebras $CT(A,B)$ and $CT(L)$ are distinct, with the exception of the comtrans algebra given by the vector triple product.

## 2. Comtrans Algebras

Let $R$ be a commutative ring with 1. Then a *comtrans algebra* over $R$ is a unital $R$- module E equipped with two trilinear operations, known as the *commutator*

$$[\ ,\ ,\ ]: E^3 \to E; (x,y,z) \mapsto [x,y,z] \tag{2.1}$$

and the *translator*

$$\langle\ ,\ ,\ \rangle: E^3 \to E; (x,y,z) \mapsto \langle x,y,z\rangle, \tag{2.2}$$

such that the commutator satisfies the *left alternative identity*

$$[x,y,z]+[y,x,z]=0, \tag{2.3}$$

the translator satisfies the *Jacobi identity*

$$\langle x,y,z\rangle+\langle y,z,x\rangle+\langle z,x,y\rangle=0, \tag{2.4}$$

and together the commutator and translator satisfy the *comtrans identity*

$$[x,y,z]+[z,y,x]=\langle x,y,z\rangle+\langle z,y,x\rangle. \tag{2.5}$$

The standard Lie algebra multiplication is the binary commutator $[x, y] = xy - yx$ of a bilinear and associative operation $(x, y) \mapsto xy$. Similarly, the standard ternary comtrans algebra operations are the ternary commutator

$$[x, y, z] = xyz - yxz \tag{2.6}$$

and translator

$$\langle x, y, z\rangle = xyz - yzx \tag{2.7}$$

of a trilinear operation

$$(x, y, z) \mapsto xyz. \tag{2.8}$$

Indeed, over a ring in which 6 is a unit, any comtrans algebra arises from the commutator (2.6) and translator (2.7) of a suitably defined trilinear operation (2.8) [5, Prop. 3.3]. The comtrans algebras $CT(A, B)$ arise from a trilinear operation on the module of rectangular matrices of given size.

**Example 2.1.** Let $R^n_m$ denote the module of $m \times n$ matrices over $R$. Fix matrices $A$ in $R^n_n$ and $B$ in $R^m_m$. Then the commutator (2.6) and translator (2.7) of the trilinear operation

$$(X, Y, Z) \mapsto XAY^TBZ \tag{2.9}$$

on $R^n_m$ make $R^n_m$ the underlying $R$-module of a comtrans algebra $CT(A, B)$.

There is also a direct construction of comtrans algebras from Lie algebras (cf. [5, Rem. 3.1(ii)]).

**Example 2.2.** Let $L$ be a Lie algebra over $R$. Then a comtrans algebra $CT(L)$ over $R$ with the same underlying $R$-module as $L$ is defined by

$$[x, y, z] = \langle x, y, z\rangle = [[x, y], z]. \tag{2.10}$$

The comtrans identity is trivial, since the commutator and translator agree. Left alternativity of the commutator follows from the skew symmetry of the Lie algebra commutator. The Jacobi identity for the translator reduces to the Jacobi identity in the Lie algebra.

Comtrans algebras of the form $CT(L)$ for abelian Lie algebras $L$, i.e. with $[x,y,z] = \langle x,y,z\rangle = 0$ for all $x,y,z$ in $L$, are described as *abelian.* If $(\mathbb{R}^3, \times)$ is the 3-dimensional real Lie algebra given by the "cross" or "vector" product of vectors, then the commutator and translator of $CT(\mathbb{R}^3, \times)$ give the traditional "vector triple product"

$$(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{y} \cdot \mathbf{z})\mathbf{x}. \tag{2.11}$$

If Example 2.2 may be viewed as giving one generalization of (2.11) (working from the left hand side), then the following gives another (working from the right hand side).

**Example 2.3.** Let $f : E \times E \to R$ be a symmetric bilinear form on an $R$-module $E$. Then a comtrans algebra $CT(E, f)$ over $R$ with the underlying $R$-module $E$ is defined by

$$[x,y,z] = \langle x,y,z\rangle = yf(x,z) - xf(y,z). \tag{2.12}$$

Verification of the identities (2.3)–(2.5) is straightforward.

## 3. Simple Comtrans and Lie Algebras

Let $E$ be a comtrans algebra over the commutative ring $R$ with 1. A submodule $J$ of (the underlying submodule of) $E$ is said to be an *ideal*, written $J \triangleleft E$, if for $j$ in $J$ and $x,y$ in $E$, all of $[j,x,y]$, $\langle j,x,y\rangle$ and $\langle y,x,j\rangle$ lie in J. Given a second comtrans algebra $D$ over $R$, a *comtrans algebra homomorphism* $f : E \to D$ is an $R$-module homomorphism $f : E \to D$ such that $[xf, yf, zf] = [x,y,z]f$ and $\langle xf, yf, zf\rangle = \langle x,y,z\rangle f$ for all $x,y,z$ in $E$. The *kernel* $Ker f$ of the comtrans algebra homomorphism $f$ is the kernel of the $R$-module homomorphism $f$.

**Proposition 3.1.** *A submodule $J$ of a comtrans algebra $E$ is an ideal of $E$ if and only if it is the kernel of a comtrans algebra homomorphism with domain $E$.*

*Proof.* Let $f : E \to D$ be a comtrans algebra homomorphism. Consider $x, y \in E$ and $j \in Kerf$. Then $[j, x, y]f = [jf, xf, yf] = [0, xf, yf] = 0$ by the trilinearity of the commutator, whence $[j, x, y] \in Kerf$. Similarly $\langle j, x, y\rangle, \langle y, x, j\rangle \in Kerf$, so that $Kerf \lhd E$.

Conversely, for $J \lhd E$, let $f : E \to E/J; x \mapsto x + J$ be the projection onto the quotient module. Define

$$[\ ,\ ,\ ] : (E/J)^3 \to E/J; (x + J, y + J, z + J) \mapsto [x, y, z] + J \tag{3.1}$$

and

$$\langle\ ,\ ,\ \rangle : (E/J)^3 \to E/J; (x + J, y + J, z + J) \mapsto \langle x, y, z\rangle + J. \tag{3.2}$$

The translator (3.2) is well-defined, since for $x - x', y - y', z - z' \in J$, one has $\langle x, y, z\rangle - \langle x', y', z'\rangle = \langle x - x', y, z\rangle + \langle x', y - y', z\rangle + \langle x', y', z - z'\rangle = \langle x - x', y, z\rangle - \langle y - y', z, x'\rangle - \langle z, x', y - y'\rangle + \langle x', y', z - z'\rangle \in J$. The commutator (3.1) is well-defined, since $[x, y, z] - [x', y', z'] = [x - x', y, z] + [x', y - y', z] + [x', y', z - z'] = [x - x', y, z] - [y - y', x'z] - [z - z', y', x'] + \langle z - z', y', x'\rangle + \langle x', y', z - z'\rangle \in J$. Together, the commutator (3.1) and translator (3.2) augment the $R$-module $E/J$ to a comtrans algebra such that $f : E \to E/J$ is a comtrans algebra homomorphism with $J = Kerf$. $\square$

The comtrans algebra $E$ is said to be *simple* if it is non-abelian, and if it has no proper non- trivial ideals (cf. [3,pp.71,290]). By Proposition 3.1, this implies that non-zero comtrans algebra homomorphisms with domain $E$ inject. One of the first research programmes in the abstract algebraic study of comtrans algebras is the classification of the simple algebras. The present paper is intended to begin such a classification. Theorem 3.2 below shows that the construction of Example 2.2

sets up a correspondence between simple Lie algebras and certain simple comtrans algebras. Let $R$-**Lie** denote the category of Lie algebras over $R$. Let $R$-**CT** denote the category of comtrans algebras over $R$. Let $CT : R\text{-}\mathbf{Lie} \to R\text{-}\mathbf{CT}$ be the functor assigning the comtrans algebra $CT(L)$ of Example 2.2 to each object $L$ of $R$-**Lie**, and with $CT(f) = f : CT(L) \to CT(K)$ for each morphism $f : L \to K$ of $R$-**Lie**.

**Theorem 3.2.** *Let $L$ be a Lie algebra over $R$. Then $L$ is simple if and only if the corresponding comtrans algebra $CT(L)$ is simple.*

*Proof.* By definition, $CT(L)$ is non-abelian if and only if $L$ is non-abelian. Suppose that $CT(L)$ is simple, but that the Lie algebra $L$ has a proper non-trivial Lie ideal $J$. Let $f : L \to L/J$ denote the projection onto the quotient Lie algebra. Then $CT(f)$ is a non-zero, non-injective comtrans algebra homomorphism with domain $CT(L)$, contradicting the simplicity of $CT(L)$. Conversely, suppose that the Lie algebra $L$ is simple, so that $[L, L] = L$. Let $J$ be an ideal of the comtrans algebra $CT(L)$. Consider $j \in J$ and $x \in L = [L, L]$, say $x = \sum_{i=1}^{n} \alpha_i[x_i, y_i]$ with $\alpha_i \in R$ and $x_i, y_i \in L$. Then $[x, j] = [\sum_{i=1}^{n} \alpha_i[x_i, y_i], j] = \sum_{i=1}^{n} \alpha_i[[x_i, y_i], j] = \sum_{i=1}^{n} \alpha_i\langle x_i, y_i, j\rangle \in J$, so that $[L, J] \subseteq J$ and the submodule $J$ of $L$ is a Lie ideal of $L$. Since $L$ is simple, it follows that $CT(L)$ is simple. $\square$

## 4. Simple $CT(A, B)$

The aim of this section is to present a simplicity criterion, namely (4.3), for the comtrans algebras $CT(A, B)$ of Example 2.1. Fix square matrices $A$ in $R_n^n$ and $B$ in $R_m^m$. The underlying module of $CT(A, B)$ is the module $R_m^n$ of $m \times n$ rectangular matrices. This module has endomorphisms

(4.1) $$R(A) : R_m^n \to R_m^n; X \mapsto XA$$

and

(4.2) $$L(B) : R_m^n \to R_m^n; X \mapsto BX.$$

**Proposition 4.1.** *The intersections* $KerR(A) \cap KerR(A^T)$ *and* $KerL(B) \cap KerL(B^T)$ *are ideals of* $CT(A,B)$.

*Proof.* Set $P = KerR(A) \cap KerR(A^T)$, a submodule of $R^n_m$. Consider $J \in P$ and $X, Y \in R^n_m$. Then $\langle J,X,Y\rangle = JAX^TBY - XAY^TBJ = -XAY^TBJ$, so that $\langle J,X,Y\rangle R(A) = -XAY^TBJA = 0$ and $\langle J,X,Y\rangle R(A^T) = -XAY^TBJA^T = 0$, whence $\langle J,X,Y\rangle \in P$. Similarly $\langle Y,X,J\rangle = YAX^TBJ - XAJ^TBY = YAX^TBJ - X(JA^T)^TBY = YAX^TBJ$, so that $\langle Y,X,J\rangle R(A) = \langle Y,X,J\rangle R(A^T) = 0$ and $\langle Y,X,J\rangle \in P$. Finally $[J,X,Y] = JAX^TBY - XAJ^TBY = 0 \in P$. Thus $P \lhd CT(A,B)$.

Now let $Q$ be the submodule $KerL(B) \cap KerL(B^T)$ of $R^n_m$. Consider $J \in Q$ and $X,Y \in R^n_m$. Then $\langle J,X,Y\rangle = JAX^TBY - XAY^TBJ = JAX^TBY$, so that $\langle J,X,Y\rangle L(B) = BJAX^TBY = 0$ and $\langle J,X,Y\rangle L(B^T) = B^TJAX^TBY = 0$, whence $\langle J,X,Y\rangle \in Q$. Also $\langle Y,X,J\rangle = YAX^TBJ - XAJ^TBY = -XA(B^TJ)^TY = 0 \in Q$. Finally $[Y,X,J] = YAX^TBJ - XAY^TBJ = 0$. Then by the comtrans identity, $[J,X,Y] = \langle J,X,Y\rangle + \langle Y,X,J\rangle - [Y,X,J] = \langle J,X,Y\rangle \in Q$. Thus $Q \lhd CT(A,B)$. □

**Proposition 4.2.** *If* $CT(A,B)$ *is simple, then:*

$$\left.\begin{array}{l} KerR(A) \cap KerR(A^T) = KerL(B) \cap KerL(B^T) = \{0\}, \\ R \quad \text{is a field, and} \quad nm > 1. \end{array}\right\} \tag{4.3}$$

*Proof.* By Proposition 4.1, the intersections are ideals. In a simple $CT(A,B)$, if either of the intersections were non-zero, then it would form all of $CT(A,B)$. But then $CT(A,B)$ would be abelian, contrary to the assumption.

If the ring $R$ has an ideal $I$, then $I^n_m \cap CT(A,B) \lhd CT(A,B)$, whence $I^n_m$ is $\{0\}$ or $R^n_m$ and $I$ is $\{0\}$ or $R$. The ring $R$ is non-zero, since $CT(A,B)$ is non- abelian. Thus $R$ is a field. Finally, $CT(A,B)$ is abelian if $nm = 1$. □

The rest of this section is devoted to a proof of the converse of Proposition 4.2. It thus proceeds under (4.3) as a general hypothesis. Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{m\times m}$. By Hypothesis (4.3),

$$\exists 1 \le S \le n.\ \exists 1 \le t \le n. \quad a_{st} \neq 0 \tag{4.4}$$

and

$$\exists 1 \le u \le m.\ \exists 1 \le v \le m. \quad b_{uv} \neq 0. \tag{4.5}$$

For indices $1 \le p \le m$ and $1 \le q \le n$, the $m \times n$ matrix whose only non-zero entry is a 1 in the intersection of the $p$-th row and the $q$-th column will be denoted by $E_{pq}$.

**Lemma 4.3.** *Under Hypothesis (4.3), consider $J \lhd CT(A,B)$. If $J \cap KerR(A)$ is non-zero, then $J = CT(A,B)$.*

*Proof.* To begin, it will be shown that

$$\exists X \in J \cap KerR(A). \quad B^T X \neq 0. \tag{4.6}$$

Consider a non-zero element $Y$ of $J \cap KerR(A)$. If $BY = 0$, then $0 \neq Y \in KerL(B)$ and (4.3) imply $Y \notin KerL(B^T)$, so $Y$ itself may serve as the $X$ in (4.6). Otherwise, $BY = \sum_{i=1}^{m}\sum_{j=1}^{n} \beta_{ij}E_{ij}$ with some $\beta_{kl} \neq 0$. Consider $X = \langle Y, E_{us}, E_{kt}\rangle = YAE_{us}^TBE_{kt} - E_{us}AE_{kt}^TBY = -E_{us}AE_{kt}^TBY \in J \cap KerR(A)$. Then $B^TX = -\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ui}a_{st}\beta_{kj}E_{ij} \neq 0$, since the entry in the $v$-th row and the $l$-th column is $-b_{uv}a_{st}\beta_{kl}$, which is non-zero by (4.5) and (4.4). This verifies (4.6).

Now fix an element $X$ as in (4.6). Note that $AX^TB \neq 0$, for otherwise $0 \neq B^TX \in KerR(A^T) \cap KerR(A)$, in contradiction to (4.3). Let $AX^TB = \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}E_{ij}$ with $\alpha_{kl} \neq 0$. For $1 \le p \le m$ and $1 \le q \le n$, one then has $-\alpha_{kl}^{-1}[X, E_{pk}, E_{lq}] = -\alpha_{kl}^{-1}XAE_{pk}^TBE_{lq} + \alpha_{kl}^{-1}E_{pk}AX^TBE_{lq} = E_{pq} \in J$, whence $J = CT(A,B)$. $\square$

**Lemma 4.4.** *Under Hypothesis (4.3), consider $J \lhd CT(A,B)$. If $J \cap KerL(B)$ is non-zero, then $J = CT(A,B)$.*

*Proof.* Define a new commutator

$$[X,Y,Z]^{\tau} = [Z,Y,X] + \langle Y,Z,X\rangle \tag{4.7}$$

and a new translator

$$\langle X,Y,Z\rangle^{\tau} = -\langle X,Z,Y\rangle \tag{4.8}$$

on $CT(A,B)$. Under these operations, the vector space $R^n_m$ becomes a comtrans algebra $CT(A,B)^{\tau}$. A subspace $J$ of $R^n_m$ is an ideal of $CT(A,B)$ if and only if it is an ideal of $CT(A,B)^{\tau}$. (The algebras $CT(A,B)$ and $CT(A,B)^{\tau}$ are "term equivalent" in the sense of universal algebra, and thus have the same congruence relations [6, p.13].) The comtrans algebras $CT(A,B)^{\tau}$ and $CT(B^T,A^T)$ are isomorphic under matrix transposition. (Thus $CT(A,B)$ and $CT(B^T,A^T)$ are "equivalent" [6, p.13].) Lemma 4.4 for $CT(A,B)$ then follows from Lemma 4.3 applied to $CT(B^T,A^T)$. $\square$

**Lemma 4.5.** *Under Hypothesis (4.3), suppose that $CT(A,B)$ contains a non-zero ideal $J$ such that*

$$J \cap KerR(A) = J \cap KerL(B) = \{0\}. \tag{4.9}$$

*Then $A$ has rank $n$.*

*Proof.* Suppose $0 \neq [c_1 \dots c_n]$ and $[c_1 \dots c_n]A = 0$ for $[c_1 \dots c_n] \in R^n_1$. Then

$$\forall X \in J, \quad AX^TB = 0. \tag{4.10}$$

For if not, say $AX^TB = \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij}E_{ij}$ with some $\alpha_{kl} \neq 0$ and $X \in J$, one would have the contradiction

$$\begin{aligned}
0 &\neq -\sum_{j=1}^{n} \alpha_{kl} c_j E_{1j} \\
&= -\sum_{j=1}^{n} c_j E_{1k} A X^T B E_{lj} \\
&= \sum_{j=1}^{n} c_j E_{lj} A E_{1k}^T B X - \sum_{j=1}^{n} c_j E_{1k} A X^T B E_{lj} \\
&= \sum_{j=1}^{n} c_j \langle E_{lj}, E_{1k}, X \rangle \in J \cap KerR(A) \\
&= \{0\}.
\end{aligned}$$

Now consider a non-zero element $Y$ of $J$. Since $J \cap KerL(B) = \{0\}$, the product $BY = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{ij} E_{ij}$ has a non-zero entry $\beta_{kl}$. Let $X = \langle E_{us}, E_{kt}, Y \rangle$, an element of $J$. By (4.10), $X = E_{us} A E_{kt}^T B Y - E_{kt} A Y^T B E_{us} = E_{us} A E_{kt}^T B Y = \sum_{j=1}^{n} a_{st} \beta_{kj} E_{uj}$, which is non-zero by (4.4). By (4.9), $0 \neq XA = \sum_{j=1}^{n} \gamma_j E_{uj}$, say with $\gamma_h \neq 0$. Then $J$ contains $[X, E_{uh}, E_{vt}] = X A E_{uh}^T B E_{vt} - E_{uh} A X^T B E_{vt} = X A E_{uh}^T B E_{vt} = \gamma_h b_{uv} E_{ut}$, where (4.10) is used in the second equality. By (4.5) and the hypothesis (4.3) that $R$ is a field, it follows that $J$ contains $E_{ut}$. But by (4.4) and (4.5), the entry in the intersection of the $s$-th row and the $v$-th column of $A E_{ut}^T B$ is non-zero, which contradicts (4.10). Thus $A$ must have full rank $n$. □

**Lemma 4.6.** *Under Hypothesis (4.3), suppose that $CT(A, B)$ contains a non-zero ideal $J$ satisfying (4.9). Then $J \not\subseteq KerL(B^T)$.*

*Proof.* Consider a non-zero element $Y$ of $J$. If $B^T Y \neq 0$, the result already holds. Otherwise, $Y^T B = 0$, while by (4.9) the product $BY = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{ij} E_{ij}$ has a non-zero entry $\beta_{kl}$. By Lemma 4.5, the matrix A is invertible. Let $X = \langle E_{u1} A^{-1}, E_{k1}, Y \rangle$, an element of $J$. Then $X = E_{u1} A^{-1} A E_{k1}^T B Y - E_{k1} A Y^T B E_{u1} A^{-1} = E_{u1} E_{k1}^T B Y =$

$\sum\limits_{j=1}^{n} \beta_{kj} E_{uj}$, and $B^T X = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} b_{ui} \beta_{kj} E_{ij}$. By (4.5), the entry of $B^T X$ in the intersection of the $v$-th row and the $l$-th column is non-zero. □

**Theorem 4.7.** *A comtrans algebra $CT(A,B)$ is simple if and only if Criterion (4.3) holds.*

*Proof.* The necessity of (4.3) for simplicity was given by Proposition 4.2. Conversely, suppose that $CT(A,B)$ satisfies (4.3). Let $J$ be a non-zero ideal of $CT(A,B)$. If $J$ violates (4.9), it is improper by Lemma 4.3 or Lemma 4.4. Suppose $J$ satisfies (4.9) and $n > 1$. By Lemma 4.5, the matrix A is invertible. By Lemma 4.6, there is an element $X$ in $J$ such that $B^T X = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \beta_{ij} E_{ij}$ has a non-zero entry $\beta_{kl}$. For $p \in \{1, \ldots, m\}$ and $q \in \{1, \ldots, n\} - \{l\}$, one has

$$\begin{aligned}
E_{pq} &= \beta_{kl}^{-1} E_{pl} X^T B E_{kq} \\
&= \beta_{kl}^{-1} X A (E_{pl} A^{-1})^T B E_{kq} - \beta_{kl}^{-1} E_{pl} A^{-1} A E_{kq}^T B X \\
&\qquad - \beta_{kl}^{-1} X A (E_{pl} A^{-1})^T B E_{kq} + \beta_{kl}^{-1} E_{pl} A^{-1} A X^T B E_{kq} \\
&= \beta_{kl}^{-1} \langle X, E_{pl} A^{-1}, E_{kq} \rangle - \beta_{kl}^{-1} [X, E_{pl} A^{-1}, E_{kq}] \in J.
\end{aligned}$$

Now fix $h \in \{1, \ldots, n\} - \{l\}$. Then $E_{uh}$ is an element of $J$ such that $B^T E_{uh} = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \alpha_{ij} E_{ij}$ has an entry $\alpha_{vh} = b_{uv}$ which is non-zero by (4.5). For $p \in \{1, \ldots, m\}$, it then follows as above that $E_{pl} = \alpha_{vh}^{-1} \langle E_{uh}, E_{ph} A^{-1}, E_{vl} \rangle - \alpha_{vh}^{-1} [E_{uh}, E_{ph} A^{-1}, E_{vl}] \in J$. Thus all the $E_{pq}$ lie in $J$, and $J$ is improper. If $J$ satisfies (4.9) and $n = 1$, then $m > 1$ by (4.3). Consider the algebra $CT(B^T, A^T)$, which also satisfies (4.3). Then $J^T$ is a non-zero ideal of $CT(B^T, A^T)$ satisfying (4.9). The preceding argument shows that $J^T$ is improper in $CT(B^T, A^T)$, so $J$ is improper in $CT(A,B)$. □

## 5. Separating Simple $CT(A,B)$ and $CT(L)$

The aim of this section is to show that, with essentially a single exception, the simple comtrans algebras $CT(L)$ given by Theorem 3.2 and $CT(A,B)$ given by

Theorem 4.7 are distinct. The critical feature of all $CT(L)$ is the agreement between their commutators and translators. The first result identifies all $CT(A,B)$ having such agreement, showing that they arise from a symmetric bilinear form according to Example 2.3. (Note that the abelian case is trivial.)

**Proposition 5.1.** *Let $A$ be an $n\times n$ matrix and $B$ an $m\times m$ matrix over a commutative ring $R$ with 1, such that the comtrans algebra $CT(A,B)$ is non-abelian, but with equal commutator and translator. Then $n=1$, and $CT(A,B)=CT(R^1_m,f)$ for the symmetric bilinear form*

$$f : R^1_m \times R^1_m ; (X,Y) \mapsto -AX^TBY. \tag{5.1}$$

*Proof.* Since $CT(A,B)$ is non-abelian, the matrix $A$ has an entry $a_{ij}$ and the matrix B has an entry $b_{kl}$ such that $a_{ij}b_{kl}\neq 0$. For $1\le p\le m$ and $1\le q\le n$, let $E_{pq}$ denote the $m\times n$ matrix whose only non-zero entry is a 1 in the intersection of the $p$-th row and the $q$-th column. If $n$ were bigger than 1, then for $j'\neq j$, $1\le j'\le n$, the entry of $[E_{kj},E_{1i},E_{lj'}]-\langle E_{kj},E_{1j},E_{lj'}\rangle = E_{1i}AE^T_{lj'}BE_{kj}-E_{1i}AE^T_{kj}BE_{lj'}$ in the intersection of the first row and the $j'$-th column, namely $-a_{ij}b_{kl}$, would be non-zero, contradicting the equality of the commutator and the translator. Thus $n=1$ and $A$ is a scalar. If the bilinear form $f$ of (5.1) were asymmetric, say with $Ab_{kl}\neq Ab_{lk}$ for some entry $b_{kl}$ of $B$, then the first entry of the column vector $[E_{k1},E_{11},E_{l1}]-\langle E_{k1},E_{11},E_{l1}\rangle = E_{11}AE^T_{l1}BE_{k1}-E_{11}AE^T_{k1}BE_{l1}$, namely $Ab_{lk}-Ab_{kl}$, would be non-zero, again contradicting the equality of the commutator and the translator. Thus the form $f$ and the $m\times m$ matrix AB are symmetric. By (2.12) and (5.1), the commutator in $CT(R^1_m,f)$ is $[X,Y,Z]=-YAX^TBZ+XAY^TBZ$, which agrees with the commutator in $CT(A,B)$. $\square$

Over an algebraically closed field of characteristic zero, the series $CT(A,B)$ and $CT(L)$ of simple comtrans algebras may now be separated, with the 3-dimensional exception of the analogue of the vector triple product (2.11), i.e. $CT(L)$ for the simple Lie algebra $L$ of type $A_1$ or $CT(A,B)$ for the matrices $A = I_1$ and $B = -I_3$.

**Theorem 5.2.** *Let $R$ be an algebraically closed field of characteristic zero. Then no simple comtrans algebra over $R$ whose underlying vector space has dimension bigger than 3 can simultaneously be of the forms $CT(A,B)$ for matrices $A, B$ and $CT(L)$ for a Lie algebra $L$ over $R$.*

*Proof.* Suppose that a simple comtrans algebra of dimension bigger than 3 is simultaneously of the forms $CT(A,B)$ and $CT(L)$. As a $CT(L)$, its commutator and translator agree. By Proposition 5.1, the commutator is then given by (2.12) for a suitable symmetric bilinear form $f$ on the underlying vector space. On the other hand, the Killing-Cartan classification of simple Lie algebras over $R$ [3, Ch. IV] associates an indecomposable simple root system with $L$ [3, Th IV.4].

Since $dim_R L > 3$, there is a pair $\alpha, \beta$ of distinct simple roots with $(\alpha, \beta) \neq 0$ [3, §§IV. 5-6]. From the corresponding root spaces $L_{\pm\alpha}, L_\beta$, pick non-zero elements $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$, and $e_\beta \in L_\beta$. Let the $\alpha$-string of roots containing $\beta$ be $\beta - r\alpha, \ldots, \beta + q\alpha$ [3, IV (18)]. Now by (2.12) and [3, IV (20)], $e_\alpha f(e_\beta, e_{-\alpha}) - e_\beta f(e_\alpha, e_{-\alpha}) = [e_\beta, e_\alpha, e_{-\alpha}] = -\frac{1}{2}q(r+1)(\alpha,\alpha)e_\beta$, whence $f(e_\alpha, e_{-\alpha}) = \frac{1}{2}q(r+1)(\alpha,\alpha)$. But by (2.12) and [3, IV (21)], $e_{-\alpha} f(e_\beta, e_\alpha) - e_\beta f(e_{-\alpha}, e_\alpha) = [e_\beta, e_{-\alpha}, e_\alpha] = -\frac{1}{2}(q+1)r(\alpha,\alpha)e_\beta$, whence $f(e_{-\alpha}, e_\alpha) = \frac{1}{2}(q+1)r(\alpha,\alpha)$. The symmetry of the form $f$ (together with the non-zero weight on $\alpha$) then gives $q = r$, which leads via $2(\alpha,\beta)/(\alpha,\alpha) = r - q$ [3, IV (18)] to the contradiction $(\alpha, \beta) = 0$. □

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# PART II. REPRESENTATION THEORY OF COMTRANS ALGEBRAS

## Abstract

Representations of a comtrans algebra are equivalent to representations of an associative universal enveloping algebra. The structure of the universal enveloping algebra is described.

## 1. Introduction

A *comtrans algebra* $E$ over a commutative ring $R$ with 1 is an $R$-module $E$ equipped with two trilinear operations, called the *commutator*

$$[\ ,\ ,\ ] : E^3 \to E;\ (x,y,z) \mapsto [x,y,z] \tag{1.1}$$

and the *translator*

$$\langle\ ,\ ,\ \rangle : E^3 \to E;\ (x,y,z) \mapsto \langle x,y,z\rangle, \tag{1.2}$$

such that the commutator satisfies *left alternativity*

$$[x,y,z] + [y,x,z] = 0, \tag{1.3}$$

the translator satisfies the *Jacobi identity*

$$\langle x,y,z\rangle + \langle y,z,x\rangle + \langle z,x,y\rangle = 0, \tag{1.4}$$

and together the commutator and translator satisfy the *comtrans identity*.

$$[x,y,z] + [z,y,x] = \langle x,y,z\rangle + \langle z,y,x\rangle. \tag{1.5}$$

Comtrans algebras were introduced in [9] as part of the solution to a problem in differential geometry [1, Problem X.3.9][5, p.16]: finding the algebraic structure in the tangent bundle that locally determines the coordinate $n$-ary loop of an $(n+1)$-web [2,§3.7]. Roughly speaking, the relationship of comtrans algebras to smooth 3-loops is analogous to the relationship of Lie algebras to Lie groups.

This paper is part of a program (cf.[6]) beginning an abstract algebraic study of comtrans algebras. The purpose here is to establish the framework for the representation theory of comtrans algebras. The categorical concept of abelian groups in the comma category of comtrans algebras over a given comtrans algebra $E$ furnishes the basic definition of $E$-modules. Using centrality theory from the universal algebraic field of Mal'cev varieties (cf.[7]), a more elementary description of $E$-modules is given in Theorem 2.9. In passing, the second section also discusses some of the details of centrality as they apply to comtrans algebras. The third section, leading up to Theorem 3.10, shows how the representation theory of a comtrans algebra $E$ is equivalent to the representation theory of an associative universal enveloping algebra $\widetilde{M}_E$ of $E$. The fourth section is devoted to the proof of Theorem 4.5, which identifies the universal enveloping algebra of a comtrans algebra $E$ over a field as the tensor algebra over $(E \wedge E) \oplus (E \otimes E) \oplus (E \otimes E)$.

## 2. Centrality

In the representation theory of algebras of various types, centrality plays an important role (often taken for granted): a module for an algebra $E$ may be defined via an extension algebra with a self-centralizing congruence furnishing the algebra $E$ as a quotient (cf.[8,3.1]). Centrality as applied to comtrans algebras will be described in this section, and related to the representation theory. The fundamental concept in centrality theory is the centralization of one congruence by another.

**Definition 2.1.** [7, 211]. Let $E$ be an algebra. Let $\beta, \gamma$ be congruences on $E$, and let $(\gamma|\beta)$ be a congruence on $\beta$. Then $\gamma$ is said *to centralize* $\beta$ by means of the *centreing congruence* $(\gamma|\beta)$ iff the following conditions are satisfied:

(C0): $(x,y)(\gamma|\beta)(x',y') \Rightarrow x\gamma x'$.
(C1): $\forall (x,y) \in \beta,\ \pi^0 : (x,y)^{(\gamma|\beta)} \longrightarrow x^\gamma;\ (x',y') \mapsto x'$ bijects.
(C2): The following three conditions are satisfied:
(RR): $\forall (x,y) \in \gamma,\ (x,x)(\gamma|\beta)(y,y)$;
(RS): $(x,y)(\gamma|\beta)(x',y') \Rightarrow (y,x)(\gamma|\beta)(y',x')$;
(RT): $(x,y)(\gamma|\beta)(x',y')$ and $(y,z)(\gamma|\beta)(y',z') \Rightarrow (x,z)(\gamma|\beta)(x',z')$.

Conditions (RR), (RS), and (RT) respectively are known as *respect for* the *reflexivity, symmetry*, and *transitivity* of $\beta$. (C2) is called *respect for equivalence.* The general conditions of Definition 2.1 reduce considerably for comtrans algebras.

**Proposition 2.2.** *Let $E$ be a comtrans algebra over a commutative ring with identity, and let $\gamma$ and $\beta$ be congruences on $E$. Then $\gamma$ centralizes $\beta$ iff for every $b \in O^\beta$, $c \in O^\gamma$ and $z \in E$, the following conditions are satisfied:*

$$[b,c,z] = 0; \tag{2.1}$$

$$\langle b,c,z\rangle = 0; \tag{2.2}$$

$$\langle z,c,b\rangle = 0; \tag{2.3}$$

$$\langle c,b,z\rangle = 0; \tag{2.4}$$

$$\langle z,b,c\rangle = 0. \tag{2.5}$$

*Proof.* We first claim that if (2.1) - (2.5) are true, then for any permutation $(X, Y, Z)$ of $(0^\beta, 0^\gamma, E)$, we have $[X, Y, Z] = \langle X, Y, Z\rangle = 0$.
This is true because of the identities:

$$[c, b, z] = -[b, c, z] = 0;$$
$$[z, c, b] = -[b, c, z] + \langle b, c, z\rangle + \langle z, c, b\rangle = 0;$$
$$[c, z, b] = -[z, c, b] = 0;$$
$$[z, b, c] = -[c, b, z] + \langle z, b, c\rangle + \langle c, b, z\rangle = 0;$$
$$\langle b, z, c\rangle = -\langle z, c, b\rangle - \langle c, b, z\rangle = 0;$$
$$\langle c, z, b\rangle = -\langle z, b, c\rangle - \langle b, c, z\rangle = 0.$$

If $\gamma$ centralizes $\beta$, then $\forall b \in 0^\beta$, $\forall c \in 0^\gamma$, $\forall z \in E$, we have:

$$\begin{array}{ccc} (b,0) & (\gamma|\beta) & (b,0) \\ (c,c) & (\gamma|\beta) & (0,0) \\ (z,z) & (\gamma|\beta) & (z,z) \\ \hline ([b,c,z],0) & (\gamma|\beta) & (0,0), \end{array}$$

whence $[b, c, z] = 0$. Equations (2.2) - (2.5) are derived similarly.

We now suppose that (2.1)-(2.5) are true. We define a relation $\eta$ on $\beta$ by $(x, y)\eta(z, t)$ iff $x - y = z - t \in 0^\beta$ and $x - z = y - t \in 0^\gamma$. It is easy to check that $\eta$ is an equivalence relation on $\beta$. To prove that $\eta$ preserves addition and scalar multiplication is straight-forward. To see that $\eta$ is a congruence on $\beta$, we only need to see that it preserves the operations of commutator and translator on $\beta$. Suppose $(x_i, y_i)\eta(z_i, t_i)$ for $1 \le i \le 3$. Since $\beta$ is a congruence on $E$, we have $[x_1, x_2, x_3] - [y_1, y_2, y_3] \in 0^\beta$ and $[z_1, z_2, z_3] - [t_1, t_2, t_3] \in 0^\beta$. Similarly, $[x_1, x_2, x_3] - [z_1, z_2, z_3] \in 0^\gamma$ and $[y_1, y_2, y_3] - [t_1, t_2, t_3] \in 0^\gamma$. Now $[x_1 - y_1, x_2, x_3] - [z_1 - t_1, z_2, z_3] = [x_1 - y_1, x_2, x_3] - [z_1 - t_1, x_2, x_3] + [z_1 - t_1, x_2 - z_2, x_3] + [z_1 - t_1, z_2, x_3 - z_3] = 0$, since $x_1 - y_1 = z_1 - t_1$, $z_1 - t_1 \in 0^\beta$, $x_2 - z_2 \in 0^\gamma$ and $x_3 - z_3 \in 0^\gamma$. The equations $[y_1, x_2 - y_2, x_3] - [t_1, z_2 - t_2, z_3] = 0$ and $[y_1, y_2, x_3 - y_3] - [t_1, t_2, z_3 - t_3] = 0$

are proved similarly. Then $[x_1,x_2,x_3]-[y_1,y_2,y_3]-[z_1,z_2,z_3]+[t_1,t_2,t_3]=[x_1-y_1,x_2,x_3]-[z_1-t_1,z_3,z_3]+[y_1,x_2-y_2,x_3]-[t_1,z_2-t_2,z_3]+[y_1,y_2,x_3-y_3]-[t_1,t_2,z_3-t_3]=0$, so that $([x_1,x_2,x_3],[y_1,y_2,y_3])\eta([z_1,z_2,z_3],[t_1,t_2,t_3])$. Similarly $(\langle x_1,x_2,x_3\rangle,\langle y_1,y_2,y_3\rangle)\eta(\langle z_1,z_2,z_3\rangle,\langle t_1,t_2,t_3\rangle)$. Therefore $\eta$ is a congruence on $\beta$. To prove that $\eta$ satisfies (C0)- (C2) is straightforward. Thus $\gamma$ centralizes $\beta$ by means of the centreing congruence $\eta$. □

A variety of algebras $E$ is called a ***Mal'cev variety*** if there is a derived operation $P$, the ***Mal'cev parallelogram***, satisfying the identities

$$(x,y,y)P=x=(y,y,x)P.$$

For comtrans algebras, one may take $(x,y,z)P=x-y+z$. Given a congruence $\gamma$ on an algebra $E$ in a Mal'cev variety, there is a unique maximal congruence $\beta$ centralized by $\gamma$ [7,228]. If $\gamma$ is the largest congruence $E\times E$ on $E$, this maximal centralized congruence $\beta$ is called the ***center congruence*** $\zeta(E)$ of $E$. The center congruence of a comtrans algebra may be characterized quite simply.

**Proposition 2.3.** *Let $E$ be a comtrans algebra, and let*

$$\begin{aligned}\beta=\{(x,y)\in E\times E|\forall z_i\in E,\quad &[x-y,z_1,z_2]=0 \text{ and}\\ &\langle x-y,z_1,z_2\rangle=0 \text{ and}\\ &\langle z_2,z_1,x-y\rangle=0\}.\end{aligned}$$

*Then $\beta$ is the center congruence on $E$.*

*Proof.* First note that if $(x,y)\in\beta$, say with $x-y=z_1$, then for $z_2,z_3\in E$ and a permutation $\pi\in\{1,2,3\}!$, we have $[z_{1\pi},z_{2\pi},z_{3\pi}]=\langle z_{1\pi},z_{2\pi},z_{3\pi}\rangle=0$. This is easy to check by using the identities for comtrans algebras. The set $\beta$ is clearly an $R$-module congruence on $E$. If $(x_i,y_i)\in\beta$ for $i=1,2,3$, then $([x_1,x_2,x_3],[y_1,y_2,y_3])\in\beta$, and $(\langle x_1,x_2,x_3\rangle,\langle y_1,y_2,y_3\rangle)\in\beta$, since $[x_1,x_2,x_3]-[y_1,y_2,y_3]=[x_1-y_1,x_2,x_3]+$

$[y_1, x_2 - y_2, x_3] + [y_1, y_2, x_3 - y_3] = 0$ and similarly $\langle x_1, x_2, x_3\rangle - \langle y_1, y_2, y_3\rangle = 0$. Thus $\beta$ is a comtrans algebra congruence on $E$. By Proposition 2.2, $E \times E$ centralizes $\beta$. Finally, we claim that $\beta$ is the maximal congruence which is centralized by $E \times E$. Suppose $\beta'$ is a congruence on $E$ centralized by $E \times E$. For $(x, y) \in \beta'$, we have $x - y \in O^{\beta'}$. By Prop. 2.2, $[x - y, z_1, z_2] = \langle x - y, z_1, z_2\rangle = \langle z_2, z_1, x - y\rangle = 0$ for all $z_1, z_2 \in E$. This implies that $(x, y) \in \beta$, hence $\beta' \leq \beta$. Thus $\beta$ is the center congruence on $E$. $\square$

The specification of the center congruence given in Proposition 2.3 will motivate the description of the universal enveloping algebra in subsequent sections.

For a commutative ring $R$ with 1, recall that an $R$-module $A$ is a set $A$ with maps $0 : \{1\} = A^\circ \to A$, $\quad - : A \to A$, $\quad + : A^2 \to A$, and $\lambda : A \to A \quad (\lambda \in R)$ such that various identities are satisfied. These identities may be expressed as the commuting of certain diagrams in the category **set** of sets.

The most general definition of module available is that of an "abelian group in the comma category over an object" [4, p.5.15]. Since this definition provides the basic conceptual framework for the representation theory of comtrans algebras, we will give an appropriate version of the definition.

**Definition 2.4.** Let $R$-**CT** (or just **CT**) be the category of comtrans algebras over a commutative ring $R$ with 1. Let $E$ be a member of $R$-**CT**. The *comma category* $R$-**CT**$/E$ (or just **CT**$/E$) of *comtrans algebras over* $E$ has as objects **CT**-morphisms $C \to E$, and as morphisms (denoted $(C \to E) \to (C^1 \to E)$ or more briefly $C \to C^1$) **CT**-morphisms $f : C \to C^1$ such that the diagram

$$\begin{array}{ccc} f : C & \longrightarrow & C^1 \\ \downarrow & & \downarrow \\ E & \xrightarrow[1_E]{} & E \end{array}$$

commutes. (Cf. [8, 3.1].)

*Remark 2.5.* There are finite products in $\mathbf{CT}/E$. The product of $\pi_1 : A_1 \to E$ and $\pi_2 : A_2 \to E$ is given by the pullback [3, §21]

$$\begin{array}{ccc} A_1 \times_E A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle \pi_2} \\ A_1 & \xrightarrow[\pi_1]{} & E \end{array}$$

i.e. $A_1 \times_E A_2 = \{(a_1, a_2) \in A_1 \times A_2 | a_1\pi_1 = a_2\pi_2\}$ with $A_1 \times_E A_2 \to E$; $(a_1, a_2) \mapsto a_1\pi_1 = a_2\pi_2$. The empty product is the terminal object of $\mathbf{CT}/E$, the object to which there is a morphism from each object in the category. This is the identity morphism $1_E : E \to E$, for from any comtrans algebra $\pi : C \to E$ over $E$ the $\mathbf{CT}$-morphism $\pi : C \to E$ is a $\mathbf{CT}/E$-morphism.

**Definition 2.6.** An $E$-module is an $R$-module in $\mathbf{CT}/E$, i.e. an object $A \to E$ of $\mathbf{CT}/E$ equipped with $\mathbf{CT}/E$-morphisms $O_E : E \to A$, $- : A \to A$, $+ : A \times_E A \to A$, and $\lambda : A \to A$ (for $\lambda \in R$) such that the $R$-module identity diagrams commute. An *$E$-morphism* or *$E$-module morphism* $f : A \to B$ between $E$- modules is a $\mathbf{CT}/E$-morphism such that $+f = (f \times_E f)+$, $-f = f-$, $O_E f = O_E$, and $\lambda f = f\lambda$ (for $\lambda \in R$). The *category* $\mathbf{A} \otimes \mathbf{CT}/E$ *of $E$-modules* has $E$-modules as its objects and $E$-morphisms between them as its morphisms.

The relationship between representation theory and centrality may now be presented. Specifically, there is an equivalence between $E$-modules and self-centralizing congruences furnishing $E$ as a quotient. This relationship holds generally in the context of Mal'cev varieties. It was described in detail [8, 3.1] in the context of quasigroups. Since the details there carry over, the results below, formulated for comtrans algebras, are stated without proof.

**Proposition 2.7.** *Let $\alpha$ be the kernel of the $\mathbf{CT}$-morphism $A \to E$ furnished by an $E$-module $A \to E$. Let $(\alpha|\alpha)$ be the kernel of the subtraction morphism $- : A \times_E A \to A$. Then $(\alpha|\alpha)$ is the (unique) centreing congruence by which $\alpha$ centralizes itself.* $\square$

**Corollary 2.8.** (a) *For $a, a'$ in $A$ with $a\alpha a'$, $(a^\alpha O_E, a)(\alpha|\alpha)(a', a+a')$.*
*(b)* $\mathbf{A}\otimes\mathbf{CT}/E$ *is a full subcategory of* $\mathbf{CT}/E$*, i.e. every* $\mathbf{CT}/E$*-morphism between* $E$*-modules is a* $E$*-morphism.* □

**Theorem 2.9.** *Let $A \to E$ be an $E$-module. Then $A \to E$ is an epimorphism in* **CT**, *say with kernel congruence $\alpha$. Identifying $A^\alpha$ with $E$ via the natural isomorphism, the* $\mathbf{CT}/E$*-objects*

$$\alpha^{(\alpha|\alpha)} \to A^\alpha \cong E; \ (a,b)^{(\alpha|\alpha)} \mapsto a^\alpha \tag{2.6}$$

*and $A \to E$ are isomorphic. Conversely, if a comtrans algebra $A$ has a self-centralizing congruence $\alpha$ for which the quotient $A^\alpha$ is (identified via a natural isomorphism with) $E$, then (2.6) is an $E$-module.* □

## 3. Comtrans modules and enveloping algebras

The aim of this section is to present the key result showing how the representation theory of a comtrans algebra $E$ is equivalent to the representation theory of an associative universal enveloping algebra $\widetilde{M}_E$. Let $E$ be a comtrans algebra over a commutative ring $R$. For $(x,y) \in E \times E$, there are three $R$-module endomorphisms defined as follows:

$$K(x,y): \quad E \to E; \ e \mapsto [e,x,y]; \tag{3.1}$$

$$R(x,y): \quad E \to E; \ e \mapsto \langle e,x,y\rangle; \tag{3.2}$$

$$L(x,y): \quad E \to E; \ e \mapsto \langle y,x,e\rangle. \tag{3.3}$$

Let $M_E$ be the subalgebra of the module endomorphism algebra $End_R E$ generated by $\{K(x,y), R(x,y), L(x,y) | x,y \in E\}$. The algebra $M_E$ is called the *enveloping* *algebra* of $E$. (The choice of (3.1)–(3.3) is motivated by Proposition 2.3.) The typical element of $M_E$ may be written as

$$\sum_{i=1}^{m} r_i C_{i1}(x_{i1}, y_{i1}) \dots C_{in_i}(x_{in_i}, y_{in_i}) \tag{3.4}$$

with $r_i \in R$, with $x_{ij}, y_{ij} \in E$, and with $C_{ij}(x_{ij}, y_{ij}) \in \{K(x_{ij}, y_{ij}),\ R(x_{ij}, y_{ij}),\ L(x_{ij}, y_{ij})\}$. A product $C_1(x_1, y_1) \dots C_n(x_n, y_n)$ will also be abbreviated as $C[(x_1, y_1), \dots, (x_n, y_n)]$ or $C[\dots, (x_n, y_n)]$ or $C$. For $e$ in $E$ and $C$ in $M_E$, there is then a comtrans word $W_C$ such that $eC[(x_1, y_1), \dots (x_n, y_n)] = W_C(e, x_1, y_1, \dots, x_n, y_n)$. Inductively, one has $W_1(e) = e$, $W_{CK(x_{n+1}, y_{n+1})}(e, x_1, y_1, \dots, x_n, y_n, x_{n+1}, y_{n+1})$ $[W_C(e, x_1, y_1, \dots, x_n, y_n), x_{n+1}, y_{n+1}]$, $W_{CR(x_{n+1}, y_{n+1})}(e, x_1, y_1, \dots,$ $x_n, y_n, x_{n+1}, y_{n+1}) = \langle W_C(e, x_1, y_1, \dots, x_n, y_n), x_{n+1}, y_{n+1}\rangle$, and $W_{CL(x_{n+1}, y_{n+1})}(e, x_1, y_1, \dots, x_n, y_n, x_{n+1}, y_{n+1})$ $= \langle y_{n+1}, x_{n+1}, W_C(e, x_1, y_1, \dots, x_n, y_n)\rangle$. A **CT**-epimorphism $f : E \to F$ induces an associative algebra epimorphism

$$M_\theta : M_E \to M_F;\ C[\dots, (x_n, y_n)] \mapsto C[\dots, (x_n\theta, y_n\theta)]. \tag{3.5}$$

The enveloping algebra construction thus leads to a functor $M.$ from the category of comtrans algebra epimorphisms to the category of associative algebras. Unfortunately, the functor $M.$ does not extend to a functor from the full category **CT** of comtrans algebras and comtrans homomorphisms to the category of associative algebras.

**Example 3.1.** Over a field $\Phi$, take $E = \Phi f$ and $F = \Phi e \oplus \Phi f'$ with $[e, f', f'] = -\langle f', e, f'\rangle = \langle e, f', f'\rangle = -\langle f', e, f'\rangle = e$ and the other basic commutators and translators being zero. Take $\theta$ as the linear mapping

$$\theta : E \to F;\ f \mapsto f'.$$

Note $\theta$ is a comtrans morphism, and $K(f,f)=0$, but $K(f,f)M_0 = K(f',f') \neq 0$. Thus $M.$ is not a functor from the category $\Phi$-**CT** to the category of associative $\Phi$-algebras. □

To overcome this failure, one may assign an $R$-algebra $\widetilde{M_E}$ to $E$ such that $M_E$ is a quotient of $\widetilde{M_E}$, and such that the assignment $\widetilde{M}.$ from the category $R$-**CT** to the category of $R$-algebras is functorial.

For an object $E$ of $R$-**CT**, let $E[X]$ be the coproduct in $R$-**CT** of $E$ with the free comtrans algebra $\langle X\rangle$ on the singleton $\{X\}$. (Note that $R$-**CT** is cocomplete [3, 32.14].)

The following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "\iota_E"] \arrow[d, "f"'] & E[X] \arrow[d, "f*g"] & \langle X\rangle \arrow[l, "\iota(x)"'] \arrow[d, "g"] & X \arrow[d] \\
E' \arrow[r, "1_{E'}"'] & E' & E' \arrow[l, "1E'"] & e'
\end{tikzcd}$$

Note $\iota_E$ embeds $E$. One may thus identify $E$ with its image $E\iota_E$ in $E[X]$. The *universal enveloping algebra* $\widetilde{M_E}$ of $E$ is then defined to be the subalgebra of $End_R E[X]$ generated by $\{K(x,y), R(x,y), L(x,y) | x,y \in E\}$. If $\theta: E \to F$ is a comtrans morphism, define $\widetilde{M_\theta}: \widetilde{M_E} \to \widetilde{M_F}$ by (3.5) and linearity.

**Proposition 3.2.** *The assignment $\widetilde{M}.$ gives a functor from $R$-**CT** to the category of $R$-algebras.*

*Proof.* Given a comtrans morphism $\theta: E \to F$, it must be shown that $\widetilde{M_\theta}: \widetilde{M_E} \to \widetilde{M_F}$ is a well-defined $R$-algebra homomorphism. To see that $\widetilde{M_\theta}$ is well-defined, suppose that $\sum_{i=1}^{m} r_i C_i = \sum_{j=1}^{m'} r'_j C'_j$ in $\widetilde{M_E}$. Then $X \sum_{i=1}^{m} r_i C_i = X \sum_{j=1}^{m'} r'_j C'_j$ implies $\sum_{i=1}^{m} r_i W_{C_i}(X,\dots) = \sum_{j=1}^{m'} r'_j W_{C'_j}(X,\dots)$. For $f \in F[X]$, one then has $f \sum_{i=1}^{m} r_i C_i \widetilde{M_\theta} = f \sum_{i=1}^{m} r_i C_i[(x_{i1}\theta, y_{i1}\theta),\dots] = \sum_{i=1}^{m} r_i W_{c_i}(f, x_{i1}\theta, y_{i1}\theta,\dots) = \sum_{i=1}^{m} r_i W_{C_i}(X,\dots)(\theta*(X \mapsto$

$f)) = \sum_{j=1}^{m'} r'_j W_{C'_j}(X,\ldots)(\theta * (X \mapsto f)) = f \sum_{j=1}^{m'} r'_j C'_j \widetilde{M}_\theta$, so that $\sum_{i=1}^{m} r_i C_i \widetilde{M}_\theta = \sum_{j=1}^{m'} r'_j C'_j \widetilde{M}_\theta$, and $\widetilde{M}_\theta$ is well-defined. The remaining verifications are straightforward. □

The first result connecting representations of $E$ and of $\widetilde{M}_E$ shows that $E$-modules furnish $\widetilde{M}_E$-modules.

**Proposition 3.3.** *Let $A \xrightarrow{\pi} E$ be an $E$-module. Then the kernel $V = \pi^{-1}(0)$ of $\pi$ is a right $\widetilde{M}_E$-module.*

*Proof.* Since the diagram

$$\begin{tikzcd} E \arrow[r, "0_E"] \arrow[d, "1_E"'] & A \arrow[d, "\pi"] \\ E \arrow[r, "1_E"'] & E \end{tikzcd}$$

commutes, $O_E$ embeds $E$. One may thus identify $E$ with its image $EO_E$ in $A$. Moreover, one can identify $\widetilde{M}_E$ with $\widetilde{M}_{EO_E}$ in $\widetilde{M}_A$. For $a, b \in A$, one has $VC(a,b) \subseteq V$, so $V\widetilde{M}_A \subseteq V$. Hence $V\widetilde{M}_E \subseteq V$, and $V$ becomes a right $\widetilde{M}_E$-module. □

Conversely, given an $\widetilde{M}_E$-module, a corresponding $E$-module will be built. To begin, note the identity

$$K(x,y) + K(y,x) - R(x,y) - R(y,x) - L(x,y) - L(y,x) = 0 \tag{3.6}$$

connecting (3.1)–(3.3), which follows from $[z,x,y] + [z,y,x] - \langle z,x,y\rangle - \langle z,y,x\rangle - \langle y,x,z\rangle - \langle x,y,z\rangle = [z,x,y] + [z,y,x] - [z,x,y] - [y,x,z] - [z,y,x] - [x,y,z] = 0$. Let $\widetilde{M}_E$-**Mod** denote the category of right $\widetilde{M}_E$-modules.

**Proposition 3.4.** *For $V \in \widetilde{M}_E$-***Mod***, there is a comtrans algebra $V \sqsupset E$ over $R$ with underlying set $V \times E$ such that:*

(a) $(v_1, e_1) + (v_2, e_2) = (v_1 + v_2, e_1 + e_2)$;

(b) $r(v,e) = (rv, re)$ *for* $r \in R$;

(c) $[(v_1,e_1),(v_2,e_2),(v_3,e_3)] = (v_1K(e_2,e_3) - v_2K(e_1,e_3) - v_3K(e_2,e_1) + v_3L(e_2,e_1) + v_3R(e_2,e_1), [e_1,e_2,e_3])$;

(d) $\langle (v_1,e_1),(v_2,e_2),(v_3,e_3)\rangle = (v_1R(e_2,e_3) - v_2R(e_3,e_1) - v_2L(e_1,e_3) + v_3L(e_2,e_1),$

$\langle e_1,e_2,e_3\rangle)$.

*Proof.* Clearly $V \sqsupset E$ is an $R$-module. We verify that $V \sqsupset E$ satisfies the identities for a comtrans algebra. For the left alternativity, $[(v_1,e_1),(v_2,e_2),(v_3,e_3)] + [(v_2,e_2),(v_1,e_1),(v_3,e_3)] = (v_1(K(e_2,e_3) - K(e_2,e_3)) + v_2(-K(e_1,e_3) + K(e_1,e_3)) + v_3(-K(e_2,e_1) - K(e_1,e_2) + R(e_2,e_1) + R(e_1,e_2) + L(e_2,e_1) + L(e_1,e_2), [e_1,e_2,e_3] + [e_2,e_1,e_3]) = (0,0)$ by (3.6). For the Jacobi identity, $\langle (v_1,e_1),(v_2,e_2),(v_3,e_3)\rangle + \langle (v_2,e_2),(v_3,e_3),(v_1,e_1)\rangle + \langle (v_3,e_3),(v_1,e_1),(v_2,e_2)\rangle = (v_1(R(e_2,e_3) - R(e_2,e_3) + L(e_3,e_2) - L(e_3,e_2)) + v_2(R(e_3,e_1) - R(e_3,e_1) + L(e_1,e_3) - L(e_1,e_3)) + v_3(L(e_2,e_1) - L(e_2,e_1) + R(e_1,e_2) - R(e_1,e_2)), \langle e_1,e_2,e_3\rangle + \langle e_2,e_3,e_1\rangle + \langle e_3,e_1,e_2\rangle) = (0,0)$. For the comtrans identity, $[(v_1,e_1),(v_2,e_2),(v_3,e_3)] + [(v_3,e_3),(v_2,e_2),(v_1,e_1)] - \langle (v_1,e_1),(v_2,e_2),(v_3,e_3)\rangle - \langle (v_3,e_3),(v_2,e_2),(v_1,e_2)\rangle = (v_1(K(e_2,e_3) - K(e_2,e_3) + L(e_2,e_3) - L(e_2,e_3) + R(e_2,e_3) - R(e_2,e_3)) + v_2(-K(e_1,e_3) - K(e_3,e_1) + R(e_1,e_3) + R(e_3,e_1) + L(e_1,e_3) + L(e_3,e_1)) + v_3(K(e_2,e_1) - K(e_2,e_1) + R(e_2,e_1) - R(e_2,e_1) + L(e_2,e_1) - L(e_2,e_1)), [e_1,e_2,e_3] + [e_3,e_2,e_1] - \langle e_1,e_2,e_3\rangle - \langle e_3,e_2,e_1\rangle) = (0,0)$ by (3.6). One may also verify that the commutator and translator are trilinear. For example, $[(v_1,e_1) + (v_2,e_2),\ (v_3,e_3),\ (v_4,e_4)] = ((v_1+v_2)K(e_3,e_4) - v_3K(e_1+e_2,e_4) - v_4K(e_3,e_1+e_2) + v_4L(e_3,e_1+e_2) + v_4R(e_3,e_1+e_2), [e_1,e_3,e_4] + [e_2,e_3,e_4]) = (v_1K(e_3,e_4) - v_3K(e_1,e_4) - v_4K(e_3,e_1) + v_4L(e_3,e_1) + v_4R(e_3,e_1) + v_2K(e_3,e_4) - v_3K(e_2,e_4) - v_4K(e_3,e_2) + v_4L(e_3,e_2) + v_4R(e_3,e_2),\ [e_1,e_3,e_4] + [e_2,e_3,e_4]) = [(v_1,e_1),\ (v_3,e_3),\ (v_4,e_4)] + [(v_2,e_2),(v_3,e_3),(v_4,e_4)]$, and for $r \in R$, one has $r[(v_1,e_1),(v_2,e_2),(v_3,e_3)] = (rv_1K(e_2,e_3) - rv_2K(e_1,e_3) - rv_3K(e_2,e_1) + rv_3L(e_2,e_1) + rv_3R(e_2,e_1), r[e_1,e_2,e_3]) = (rv_1K(e_2,e_3) - v_2K(re_1,e_3) - v_3K(e_2,re_1) +$

$v_3L(e_2, re_1) - v_3R(e_2, re_1), [re_1, e_2, e_3]) = [r(v_1, e_1), (v_2, e_2), (v_3, e_3)]$. Thus $V \sqsupset E$ is a comtrans algebra over $R$. $\square$

Define $\pi : V \sqsupset E \to E;\ (v, e) \mapsto e$. Note that $\pi$ is a $\mathbf{C}T$-morphism. Thus $(\pi : V \sqsupset E \to E) \in \mathbf{C}T/E$.

**Proposition 3.5.** *If $V$ is an $\widetilde{M}_E$-module, then $\pi : V \sqsupset E \to E$ is an $E$-module.*

*Proof.* Let $\alpha$ be the kernel congruence of $\pi : V \sqsupset E \to E$. Define $(\alpha|\alpha)$ on $\alpha$ by $((v_1, e), (v_2, e))(\alpha|\alpha)((v_3, e), (v_4, e)) \Leftrightarrow v_1 - v_2 = v_3 - v_4$. Then $\alpha$ is self-centralizing by $(\alpha|\alpha)$. By Theorem 2.9, (2.6) is an $E$-module, and there is an isomorphism $\alpha^{(\alpha|\alpha)} \to V \sqsupset E; ((v_1, e), (v_2, e))^{(\alpha|\alpha)} \mapsto (v_1 - v_2, e)$. $\square$

The passages from $E$-modules to $\widetilde{M}_E$-modules and back given by Propositions 3.3 and 3.5 will now be extended to functors $T : \mathbf{A} \otimes \mathbf{C}T/E \to \widetilde{M}_E$-**Mod** and $S : \widetilde{M}_E$-**Mod**$\to \mathbf{A} \otimes \mathbf{C}T/E$ that ultimately (Theorem 3.10) yield a category equivalence. For an $E$-module $\pi : A \to E$, define $T(\pi : A \to E)$ to be the $\widetilde{M}_E$-module $V = \pi^{-1}(0)$ given by Proposition 3.3. For an $E$-module morphism

$$\begin{array}{ccc} A_1 & \xrightarrow{\theta} & A_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ E & \xrightarrow[1_E]{} & E, \end{array}$$

define $T(\theta) : \pi_1^{-1}(0) \to \pi_2^{-1}(0)$ to be the restriction of $\theta$.

**Proposition 3.6.** *$T : \mathbf{A} \otimes \mathbf{C}T/E \to \widetilde{M}_E$- **Mod** is a functor.*

*Proof.* The main thing to check is that $T(\theta)$ is a well defined right $\widetilde{M}_E$-module morphism. For $V \in \pi_1^{-1}(0)$, $V\theta\pi_2 = V\pi_1 = 0$, so $T(\theta)$ is well defined. It is clearly an $R$-module homomorphism. It remains to verify

$$(vC)T(\theta) = (vT(\theta))C \tag{3.7}$$

for $v \in \pi_1^{-1}(0)$ and $C \in \widetilde{m}_E$. This will be done inductively on the length $n$ of $C = C[(x_1, y_1), \ldots, (x_n, y_n)]$. If $n = 0$, (3.7) is immediate. Suppose that (3.7) holds for a fixed $n$, and $C = C[(_1, y_1), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1})] = D[(x_1, y_1), \ldots, (x_n, y_n)] \circ \widetilde{M}$. If $\widetilde{m}K(x_{n+1}, y_{n+1})$, then $(vC)T(\theta) = (vC)\theta = [vD \circ \widetilde{m}]\theta = [vD, x_{n+1}, y_{n+1}]\theta = [vD\theta, x_{n+1}\theta, y_{n+1}\theta] = [v\theta D, x_{n+1}\theta - x_{n+1} + x_{n+1}, y_{n+1}\theta - y_{n+1} + y_{n+1}]$ by Proposition 2.7, identifying $E$ with its images under $0 : E \to A_1$ and $0 : E \to A_2$. For $e \in E$, one has $e = e\pi_1 = e\theta\pi_2$ and $e = e\pi_2$. This implies $(e\theta - e)\pi_2 = 0$, so $e\theta - e \in \pi_2^{-1}(0)$. Thus $(vC)T(\theta) = [v\theta D, x_{n+1}, y_{n+1}] + [v\theta D, x_{n+1}\theta - x_{n+1}, y_{n+1}] + [v\theta D, x_{n+1}\theta - x_{n+1}, y_{n+1}\theta - y_{n+1}] + [v\theta D, x_{n+1}, y_{n+1}\theta - y_{n+1}] = [v\theta D, x_{n+1}, y_{n+1}] = v\theta D \circ \widetilde{m} = v\theta C = (vT(\theta))C$.
For $\widetilde{m} = R(x_{n+1}, y_{n+1})$ and $\widetilde{m} = L(x_{n+1}, y_{n+1})$, the proofs are similar. The remaining functoriality properties are straightforward. □

For an $\widetilde{M}_E$-module $V$, define $S(V)$ to be the $E$-module $\pi : V \sqsupset E \to E$ given by Proposition 3.5. For an $\widetilde{M}_E$-module morphism $\theta : v_1 \to v_2$, define $S(\theta) : V_1 \sqsupset E \to V_2 \sqsupset E$; $(v, e) \mapsto (v\theta, e)$.

**Proposition 3.7.** *$S : \widetilde{M}_E$-***Mod**$\to$ $\mathbf{A} \otimes \mathbf{CT}/E$ *is a functor.*

*Proof.* The main thing to check is that $S(\theta)$ is an $E$-module morphism. It is clearly an $R$-module homomorphism. For $(v_1, e_1), (v_2, e_2), (v_3, e_3) \in v_1 \sqsupset E$,
$[(v_1, e_1), (v_2, e_2), (v_3, e_3)]S(\theta) = (v_1K(e_2, e_3)\theta - v_2K(e_1, e_3)\theta - v_3K(e_2, e_1)\theta + v_3L(e_2, e_1)\theta + v_3R(e_2, e_1)\theta, [e_1, e_2, e_3]) = (v_1\theta K(e_2, e_3) - v_2\theta K(e_1, e_3) - v_3\theta K(e_2, e_1) + v_3\theta L(e_2, e_1) + v_3\theta R(e_2, e_1), [e_1, e_2, e_3]) = [(v_1\theta, e_1), (v_2\theta, e_2), (v_3\theta, e_3)] = [(v_1, e_1)S(\theta), (v_2, e_2)S(\theta), (v_3, e_3)S(\theta)]$. The equation $\langle (v_1, e_1), (v_2, e_2), (v_3, e_3) \rangle S(\theta) = \langle (v_1, e_2)S(\theta), (v_2, e_2)S(\theta), (v_3, e_3)S(\theta) \rangle$ is proved similarly. The diagram

$$\begin{array}{ccc} V_1 \sqsupset E & \xrightarrow{S(\theta)} & V_2 \sqsupset E \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ E & \xrightarrow[1_E]{} & E \end{array}$$

commutes. Thus $S(\theta)$ is a **CT**/$E$-morphism. By Corollary 2.8, it follows that $S(\theta)$ is an $E$-module morphism. The remaining functoriality properties are straightforward. $\square$

**Lemma 3.8.** *For each $\widetilde{M}_E$-module $V$, there is a natural $\widetilde{M}_E$-module isomorphism*

$$h_V : T \circ S(V) \to V;\ (v, 0) \mapsto v. \tag{3.8}$$

*Proof.* It is straightforward to see that $h_V$ is a natural $R$-module isomorphism. For $C = C[(x_1, y_1), \ldots, (x_n, y_n)] \in \widetilde{M}_E$, it will be proved inductively on the length $n$ of $C$ that

$$(v, 0)C = (vC, 0) \tag{3.9}$$

for $v \in V$. If $n = 0$, (3.9) is immediate. Suppose that (3.9) holds for a fixed $n$, and $C = C[(x_1, y_1), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1})] = D[\ldots (x_n, y_n)] \circ \widetilde{m}$. If $\widetilde{m} = R(x_{n+1}, y_{n+1})$, then $(v, 0)C = [(v, 0)D]\widetilde{m} = (vD, 0)R(x_{n+1}, y_{n+1}) = \langle (vD, 0), (0, x_{n+1}), (0, y_{n+1}) \rangle = (vDR(x_{n+1}, y_{n+1}), 0) = (vC, 0)$. If $\widetilde{m} = K(x_{n+1}, y_{n+1})$ or $\widetilde{m} = L(x_{n+1}, y_{n+1})$, $(v, 0)C = (vC, 0)$ is proved similarly. Thus $h_V$ is an $\widetilde{M}_E$-module isomorphism. $\square$

**Lemma 3.9.** *For each $E$-module $\pi : A \to E$, there is a natural $E$-module isomorphism*

$$g_A : S \circ T(A) \to A;\ (v, e) \mapsto v + e. \tag{3.10}$$

*Proof.* Note $S \circ T(A) = \pi^{-1}(0) \sqsupset E$, and $E$ is identified with its image in $A$ under $0 : E \to A$. Now $g_A$ injects, since $v + e = 0 \Rightarrow e\pi = v\pi + e\pi = (v + e)\pi = 0$, whence $e = 0$ and then $v = 0$. Also $g_A$ surjects, since for $a \in A$, one has $a = (a - a\pi, a\pi)g_A$. It remains to be shown that $g_A$ is an $E$-module morphism. By Corollary 2.8, it suffices to show that $g_A$ is a $\mathbf{CT}/E$-morphism. Clearly $g_A$ is an $R$- module morphism. For $(v_1, e_1), (v_2, e_2), (v_3, e_3) \in \pi^{-1}(0) \sqsupset E$, one has $[(v_1, e_1), (v_2, e_2), (v_3, e_3)]g_A = v_1K(e_2, e_3) - v_2K(e_1, e_3) - v_3K(e_2, e_1) + v_3L(e_2, e_1) + v_3R(e_2, e_1) + [e_1, e_2, e_3] = [v_1, e_2, e_3] + [e_1, v_2, e_3] + [e_1, e_2, v_3] + [e_1, e_2, e_3] = [v_1 + e_1, v_2 + e_2, v_3 + e_3] = [(v_1, e_1)g_A, (v_2, e_2)g_A, (v_3, e_3)g_A]$, the penultimate equality holding since $ker(\pi : A \to E)$ is self-centralizing. The equation $\langle (v_1, e_1), (v_2, e_2), (v_3, e_3) \rangle g_A = \langle (v_1, e_1)g_A, (v_2, e_2)g_A, (v_3, e_3)g_A \rangle$ is proved similarly. The diagram

$$\begin{tikzcd}
\pi_A^{-1}(0) \sqsupset E \arrow[r, "J_A"] \arrow[d] & A \arrow[d] \\
E \arrow[r, "1_E"'] & E
\end{tikzcd}$$

commutes, since $(v, e)g_A\pi = (v + e)\pi = v\pi + e\pi = e\pi$. Thus $g_A$ is a $\mathbf{CT}/E$-morphism, as required. $\square$

**Theorem 3.10.** *The categories* $\mathbf{A} \otimes \mathbf{CT}/E$ *of* $E$*-modules and* $\widetilde{M}_E$**-Mod** *of* $\widetilde{M}_E$*-modules are equivalent via the functors* $S$ *and* $T$*.*

*Proof.* Given an $\mathbf{A} \otimes \mathbf{CT}/E$-morphism $\theta : A \to A'$, the construction (3.10) gives a commutative diagram

$$\begin{tikzcd}
\pi_{-1}(0) \sqsupset E \arrow[r, "S\circ T(\theta)"] \arrow[d, "g_A"'] & \pi^{-1}(0) \sqsupset E \arrow[d, "g_{A'}"] \\
A \arrow[r, "\theta"] & A'
\end{tikzcd} .$$

Given an $\widetilde{M}_E$-**Mod**-morphism $\phi : V \to V'$, the construction (3.8) gives a commutative diagram

$$\begin{tikzcd}
\pi^{-1}(0) \arrow[r, "T\circ S(\phi)"] \arrow[d, "h_V"'] & \pi^{-1}(0) \arrow[d, "h_{V'}"] \\
V \arrow[r, "\phi"] & V'
\end{tikzcd}$$

Thus $S \circ T \cong 1_{\mathbf{A}\otimes\mathbf{C}T/E}$ and $T \circ S \cong 1_{\widetilde{M}_E-\mathbf{Mod}}$. $\square$

## 4. Structure of the universal enveloping algebra

The universal enveloping algebra of a comtrans algebra was defined somewhat abstractly in the previous section. In this section, an explicit description of the universal enveloping algbebra as a certain tensor algebra will be given (Theorem 4.5).

Let $E$ be a comtrans algebra over a field $\Phi$. Let the underlying vector space of $E$ have basis $\{e_j | j \in J\}$, where the index set $J$ is totally ordered. Let $S = \{k(e_i, e_j), r(e_p, e_q), \ell(e_p, e_1) | i < j; i, j, p, q \in J\}$, and let $V$ be a vector space over $\Phi$ with basis $S$. By definition, the tensor algebra $T(V)$ based on $V$ is of the form

(4.1) $$T(V) = \Phi \oplus V_1 \oplus V_2 \oplus \ \ldots \ \oplus V_n \oplus \ldots\ ,$$

where $V_1 = V$ and $V_n = \otimes^n V$. For $\alpha_i, \beta_j \in \Phi$, define:

(4.2) $$r(\textstyle\sum \alpha_i e_i, \sum \beta_j e_j) = \sum_{i,j} \alpha_i \beta_j \gamma(e_i, e_j),$$

(4.3) $$\ell(\textstyle\sum \alpha_i e_i, \sum \beta_j, e_j) = \sum_{i,j} \alpha_i \beta_j \ell(e_i, e_j).$$

(4.4)

For $i > j$, define $k(e_i, e_j) = -k(e_j, e_i) + r(e_i, e_j) + r(e_j, e_i) + \ell(e_i, e_j) + \ell(e_j, e_i)$.

For $i = j$, define

(4.5) $$k(e_i, e_i) = r(e_i, e_i) + \ell(e_i, e_i)$$

Then for $\alpha_i, \beta_j \in \Phi$, define

(4.6) $$k(\textstyle\sum \alpha_i e_i, \sum \beta_j e_j) = \sum_{i,j} \alpha_i \beta_j k(e_i, e_j).$$

**Lemma 4.1.** *For any $a, b \in E$,*

$$-k(a,b) - k(b,a) + r(a,b) + r(b,a) + \ell(a,b) + \ell(b,a) = 0. \tag{4.7}$$

*Proof.* Let $a = \sum \alpha_i e_i$ and $b = \sum \beta_j e_j$ with $\alpha_i, \beta_j \in \Phi$. By [4.2], [4.3] and [4.6], the expression on the left of [4.7] is

$$\begin{aligned}
&-\sum_{i,j} \alpha_i\beta_j k(e_i,e_j) - \sum_{i,j} \alpha_i\beta_j k(e_j,e_i) + \sum_{i,j} \alpha_i\beta_j r(e_i,e_j) + \sum_{i,j} \alpha_i\beta_j r(e_j,e_i) \\
&+\sum_{i,j} \alpha_i\beta_j \ell(e_i,e_j) + \sum_{i,j} \alpha_i\beta_j \ell(e_j,e_i) = -\sum_{i\neq j} \alpha_i\beta_j k(e_i,e_j) - \sum_{i\neq j} \alpha_i\beta_j k(e_j,e_i) \\
&+\sum_{i\neq j} \alpha_i\beta_j r(e_i,e_j) + \sum_{i\neq j} \alpha_i\beta_j r(e_j,e_i) + \sum_{i\neq j} \alpha_i\beta_j \ell(e_i,e_j) + \sum_{i\neq j} \alpha_i\beta_j \ell(e_j,e_i) \\
&-\sum_{i=j} \alpha_i\beta_j k(e_i,e_j) - \sum_{i=j} \alpha_i\beta_j k(e_j,e_i) + \sum_{i=j} \alpha_i\beta_j r(e_i,e_j) + \sum_{i=j} \alpha_i\beta_j r(e_j,e_i) \\
&+\sum_{i=j} \alpha_i\beta_i \ell(e_i,e_j) + \sum_{i=j} \alpha_i\beta_j \ell(e_j,e_i) = \sum_{i\neq j} \alpha_i\beta_j(-k(e_i,e_j) - k(e_j,e_i) + r(e_i,e_j) \\
&+r(e_j,e_i) + \ell(e_i,e_j) + \ell(e_j,e_i)) + \sum_{i=j} \alpha_i\beta_j(-k(e_i,e_j) + r(e_i,e_j) + \ell(e_i,e_j)) \\
&+\sum_{i=j} \alpha_i\beta_j(-k(e_j,e_i) + r(e_j,e_i) + \ell(e_j,e_i)),
\end{aligned}$$

which reduces to 0 by [4.4] and [4.5]. □

**Lemma 4.2.** *There is a comtrans algebra $T(V) \sqsupset E$ over $\Phi$ with underlying set $T(V) \times E$ and:*

(a) $(t_1, a_1) + (t_2, a_2) = (t_1 + t_2, a_1 + a_2)$;

(b) $\alpha(t_1, a_1) = (\alpha t_1, \alpha a_2)$ *for* $\alpha \in \Phi$;

(c) $[(t_1,a_1),(t_2,a_2),(t_3,a_3)] = (t_1 \otimes k(a_2,a_3) - t_2 \otimes k(a_1,a_3) - t_3 \otimes k(a_2,a_1)$
$+t_3 \otimes \ell(a_2,a_1) + t_3 \otimes r(a_2,a_1), [a_1,a_2,a_3])$;

(d) $\langle (t_1,a_1),(t_2,a_2),(t_3,a_3)\rangle = (t_1 \otimes \gamma(a_2,a_3) - t_2 \otimes r(a_3,a_1) - t_2 \otimes \ell(a_1,a_3)$
$+t_3 \otimes \ell(a_2,a_1), \langle a_1,a_2,a_3\rangle)$.

*Proof.* Clearly, $T(V) \sqsupset E$ is a vector space over $\Phi$. We verify that $T(V) \sqsupset E$ satisfies the identities for a comtrans algebra. For the left alternativity,

$[(t_1,a_1),(t_2,a_2),(t_3,a_3)] + [(t_2,a_2),(t_1,a_1),(t_3,a_3)]$
$= (t_1 \otimes (k(a_2,a_3) - k(a_2,a_3) + t_2 \otimes (-k(a_1,a_3) + k(a_1,a_3))$

$+t_3 \otimes (-k(a_2, a_1) - k(a_1, a_2) + r(a_2, a_1) + (a_1, a_2)$
$+\ell(a_2, a_1) + \ell(a_1, a_2)), [a_1, a_2, a_3] + [a_2, a_1, a_3]) = (0, 0)$ by Lemma 4.1. For the Jacobi identity, $\langle(t_1, a_1), (t_2, a_2), (t_3, a_3)\rangle + \langle(t_2, a_2), (t_3, a_3), (t_1, a_1)\rangle$
$+\langle(t_3, a_3), (t_1, a_1), (t_2, a_2)\rangle = (t_1 \otimes (r(a_2, a_3) - r(a_2, a_3) + \ell(a_3, a_2) - \ell(a_3, a_2))$
$+t_2 \otimes (r(a_3, a_1) - r(a_3, a_1) + \ell(a_1, a_3) - \ell(a_1, a_3))$
$+t_3 \otimes (\ell(a_2, a_1) - \ell(a_2, a_1) + r(a_1, a_2) - r(a_1, a_2)),$
$\langle a_1, a_2, a_3\rangle + \langle a_2, a_3, a_1\rangle + \langle a_3, a_1, a_2\rangle) = (0, 0)$. For the comtrans identity,
$[(t_1, a_1), (t_2, a_2), (t_3, a_3)] + [(t_3, a_3), (t_2, a_2), (t_1, a_1)]$
$-\langle(t_1, a_1), (t_2, a_2), (t_3, a_3)\rangle - \langle(t_3, a_3), (t_2, a_2), (t_1, a_1)\rangle$
$= (t_1 \otimes (k(a_2, a_3) - k(a_2, a_3) + \ell(a_2, a_3) - \ell(a_2, a_3)$
$+r(a_2, a_3) - r(a_2, a_3)) + t_2 \otimes (-k(a_1, a_3) - k(a_3, a_1)$
$+r(a_1, a_3) + r(a_3, a_1) + \ell(a_1, a_3) + \ell(a_3, a_1))$
$+t_3 \otimes (k(a_2, a_1) - k(a_2, a_1) + r(a_2, a_1) - r(a_2, a_1)$
$+\ell(a_2, a_1) - \ell(a_2, a_1)), [a_1, a_2, a_3] + [a_3, a_2, a_1] - \langle a_1, a_2, a_3\rangle - \langle a_3, a_2, a_1\rangle) = (0, 0)$ by Lemma 4.1. Finally, we may verify that the commutator and translator are trilinear. For example, $[(t_1, a_1) + (t_2, a_2), (t_3, a_3), (t_4, a_4)]$
$= ((t_1 + t_2) \otimes k(a_3, a_4) - t_3 \otimes k(a_1 + a_2, a_4) - t_4 \otimes k(a_3, a_1 + a_2) + t_4 \otimes \ell(a_3, a_1 + a_2) + t_4 \otimes r(a_3, a_1 + a_2), [a_1, a_3, a_4] + [a_2, a_3, a_4])$
$= (t_1 \otimes k(a_3, a_4) - t_3 \otimes k(a_1, a_4) - t_4 \otimes k(a_3, a_1) + t_4 \otimes \ell(a_3, a_1) + t_4 \otimes r(a_3, a_1) + t_2 \otimes k(a_3, a_4) - t_3 \otimes k(a_2, a_4) - t_4 \otimes k(a_3, a_2) + t_4 \otimes \ell(a_3, a_2)$
$+t_4 \otimes r(a_3, a_2), [a_1, a_3, a_4] + [a_2, a_3, a_4]) = [(t_1, a_1), (t_3, a_3), (t_4, a_4)]+$
$[(t_2, a_2), (t_3, a_3), (t_4, a_4)]$, while for $\alpha \in \Phi$, one has $\alpha[(t_1, a_1), (t_2, a_2), (t_3, a_3)] =$
$(\alpha t_1 \otimes k(a_2, a_3) - \alpha t_2 \otimes k(a_1, a_3) - \alpha t_3 \otimes k(a_2, a_1) + \alpha t_3 \otimes \ell(a_2, a_1) +$
$\alpha t_3 \otimes r(a_2, a_1), \alpha[a_1, a_2, a_3]) = (\alpha t_1 \otimes k(a_2, a_3) - t_2 \otimes k(\alpha a_1, a_3) - t_3 \otimes k(a_2, \alpha a_1) +$
$t_3 \otimes \ell(a_2, \alpha a_1) - t_3 \otimes r(a_2, \alpha a_1), [\alpha a_1, a_2, a_3]) = [(\alpha t_1, \alpha a_1), (t_2, a_2), (t_3, a_3)]$. Similarly one can show that the commutator is linear in its second and third arguments, and that the translator is trilinear. Thus $T(V) \sqsupset E$ is a comtrans algebra over

$\phi$ □

Define $\Delta : E \to T(V) \sqsupset E$; $a \mapsto (0, a)$. Note that $\Delta$ is a comtrans morphism, and that the following diagram commutes:

$$
\begin{tikzcd}
E \arrow[r] \arrow[d, "\Delta"'] & E[X] \arrow[d, "\Delta * g"] & \langle X \rangle \arrow[l] \arrow[d, "g"] & X \arrow[d] \\
T(V) \sqsupset E \arrow[r] & T(V) \sqsupset E & T(V) \sqsupset E(1,0) \arrow[l, "1"'] &
\end{tikzcd} \tag{4.8}
$$

**Lemma 4.3.** *Let* $\Sigma = \{K(e_i, e_j), R(e_p, e_q), L(e_p, e_q) | i < j; i, j, p, q \in J\}$. *The function* $f : \Sigma \to S; K(e_i, e_j) \mapsto k(e_i, e_j), R(e_p, e_q) \mapsto r(e_p, e_q), L(e_p, e_q) \mapsto \ell(e_p, e_q)$ *is well defined.*

*Proof.* Suppose $K(e_{i_1}, e_{j_1}) = K(e_{i_2}, e_{j_2}) \in \Sigma$. By (4.8), $XK(e_{i_1}, e_{j_1})\Delta * g = XK(e_{i_2}, e_{j_2})$
$\Delta * g \Rightarrow [X, e_{i_1}, e_{j_1}]\Delta * g = [X, e_{i_2}, e_{j_2}]\Delta * g$
$\Rightarrow [(1,0), (0, e_{i_1}), (0, e_{j_1})] = [(1,0), (0, e_{i_2}), (0, e_{j_2})]$
$\Rightarrow (k(e_{i_1}, e_{j_1}), 0) = (k(e_{i_2}, e_{j_2}), 0)$
$\Rightarrow k(e_{i_1}, e_{j_1}) = k(e_{i_2}, e_{j_2})$. Using the same argument, one can show that for any $C(e_{i_1}, e_{j_1}) = D(e_{i_2}, e_{j_2}) \in \Sigma$, one has $C(e_{i_1}, e_{j_1})f = D(e_{i_2}, e_{j_2})f$, so that $f$ is well-defined. □

**Lemma 4.4.** *If* $\widetilde{U}_1 \circ \cdots \circ \widetilde{U}_n \in \widetilde{M}_E$ *such that for* $m \neq 0, \widetilde{U}_m \in \Sigma$, *then* $X\widetilde{U}_1 \circ \cdots \circ \widetilde{U}_n(\Delta * g) = (\widetilde{U}_1 f \otimes \cdots \otimes \widetilde{U}_n f, 0)$ *with the map* $f$ *of Lemma 4.3.*

*Proof.* For $n = 0$, one has $X1_{E[\times]}(\Delta * g) = X(\Delta * g) = (1, 0)$. Suppose that the lemma holds for products of at most $n$ factors. For $\widetilde{U}_{n+1} = K(e_i, e_j)$, one has $X\widetilde{U}_1 \circ \cdots \circ \widetilde{U}_n \circ \widetilde{U}_{n+1}(\Delta * g) = [X\widetilde{U}_1 \circ \cdots \circ \widetilde{U}_n, e_i, e_j]\Delta * g = [X\widetilde{U}_1 \circ \cdots \circ \widetilde{U}_n \Delta * g,\ e_i \Delta * g, e_j \Delta * g] = [(\widetilde{U}_1 f \otimes \cdots \otimes \widetilde{U}_n f, 0), (0, e_i), (0, e_j)] = (\widetilde{U}_1 f \otimes \cdots \otimes \widetilde{U}_n f \otimes k(e_i, e_j), 0) = (\widetilde{U}_1 f \otimes \cdots \otimes \widetilde{U}_n f \otimes \widetilde{U}_{n+1} f, 0)$. For $\widetilde{U}_{n+1} = R(e_p, e_q)$ or $\widetilde{U}_{n+1} = L(e_p, e_q)$, the proof is similar. □

**Theorem 4.5.** *Let $E$ be a comtrans algebra over a field $\Phi$. Let*

$$V = (E \wedge E) \oplus (E \otimes E) \oplus (E \otimes E).$$

*Then the universal enveloping algebra $\widetilde{M_E}$ of $E$ is isomorphic to the tensor algebra $T(V)$ over $V$.*

*Proof.* With suitable identifications, the set $\Sigma$ of Lemma 4.3 is a basis for $V$. Let $\Sigma^*$ be the free monoid over $\Sigma$. The embedding of $\Sigma$ into $\widetilde{M_E}$ extends to a monoid homomorphism of $\Sigma^*$ into (the underlying monoid of) $\widetilde{M_E}$. By Lemma 4.4, this homomorphism injects. Identify $\Sigma^*$ with its image. The image $\Sigma^*$ spans $\widetilde{M_E}$, since $K(e_i, e_i) = R(e_i, e_i) + L(e_i, e_i)$ and $K(e_j, e_i) = -K(e_i, e_j) + R(e_i, e_j) + R(e_j, e_i) + L(e_i, e_j) + L(e_j, e_i)$ for $i < j$. By Lemma 4.4, the image $\Sigma^*$ is linearly independent. Thus $\Sigma^*$ is a basis for (the underlying vector space of) $\widetilde{M_E}$, whence $\widetilde{M_E}$ is isomorphic to the tensor algebra $T(V)$ over $V$. $\square$

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# PART III. COMTRANS ALGEBRAS AND BILINEAR FORMS

## 1. Introduction

Comtrans algebras were introduced [9] in answer to a problem from differential geometry [2, Problem X.3.9][6, p.16]: finding the algebraic structure in the tangent bundle corresponding to the coordinate $n$-ary loop of an $(n+1)$-web [3,§3.7]. The algebraic structure consists of a system of comtrans algebras, interlaced with some of the "$W$-algebras" (now known as Akivis algebras) that had been introduced earlier by Akivis in correspondence with the coordinate binary loops of 3-webs [1] [2,§IX.6] [4]. The current paper is part of a progamme (cf. [7], [8]) beginning a study of comtrans algebras from a purely algebraic point of view. It was noted in [9, Remark 3.1 (ii)] that a comtrans algebra arises from the repeated commutator [[ , ], ] of a Lie algebra. For the Lie algebra of Euclidean space $\mathbb{R}^3$ under the "vector" or "cross" product $\times$, this repeated product is the "vector triple product"

$$(\underset{\sim}{x} \times \underset{\sim}{y}) \times \underset{\sim}{z} = \underset{\sim}{y}(\underset{\sim}{x} \cdot \underset{\sim}{z}) - \underset{\sim}{x}(\underset{\sim}{y} \cdot \underset{\sim}{z}). \tag{1.1}$$

One could thus regard the vector triple product comtrans algebra as arising from the Euclidean inner product on $\mathbb{R}^3$ according to the right hand side of (1.1), rather than from the repeated Lie algebra commutator appearing on the left. The main topic of the present paper is a generalization (3.1-2) of this construction, producing a comtrans algebra $CT(E, \beta)$ from a pair $(E, \beta)$ consisting of a unital module $E$ over a commutative ring $R$ with 1 and a bilinear form $\beta : E^2 \to R$. A "transposed" comtrans algebra $CT(E, \beta)^\tau$ is also given by the pair $(E, \beta)$ (3.3-4). These constructions compare with the currently popular methods of making algebras out of spaces with forms, such as Jordan algebras or Clifford algebras. One major advantage of the comtrans algebras $CT(E, \beta)$ and $CT(E, \beta)^{\tau}$ is that they do not require any extension of the underlying module $E$ in order to achieve closure. By contrast, the underlying modules of Clifford algebras (for example) blow up exponentially in size.

Section 2 presents the definition of comtrans algebras (2.1-5), and covers some elementary topics that are needed, such as the notions of ideal, abelian algebras, and simple algebras. The transposition relationship between a pair of comtrans algebras, typified by $CT(E, \beta)$ and $CT(E, \beta)^{\tau}$, is also described (2.6-7). The third section gives the basic construction of the comtrans algebras $CT(E, \beta)$ and $CT(E, \beta)^{\tau}$ from a module $(E, \beta)$ with a bilinear form. For tight connections between the form and the algebras, some restrictions on the underlying module $E$ are required. Appropriate restrictions are encoded in the concept of "formed space" (Definition 3.3), making the underlying module free of rank more than 1. Theorem 3.4 shows how simplicity of the comtrans algebras is equivalent to non-degeneracy of the form and simplicity of the ring of scalars. In general, the radical of a bilinear form on a formed space may be described in comtrans algebra terms (Proposition 3.5). Theorem 3.6 and its corollary show that the automorphism groups of the formed space $(E, \beta)$ and of the comtrans algebras $CT(E, \beta)$ and $CT(E, \beta)^{\tau}$ coincide. The fourth section is concerned with the problem of recognizing when a comtrans algebra is a "form algebra", i.e $CT(E, \beta)$ or $CT(E, \beta)^{\tau}$ for a formed space $(E, \beta)$. The answer is given by Theorem 4.1. Consideration of the hyperbolic plane (Example 4.2) shows that the two-dimensional case is anomalous.

## 2. Comtrans algebras

Let $R$ be a commutative ring with 1. A *comtrans algebra* over $R$ is a unital $R$-module $E$ equipped with a trilinear operation.

$$[\ ,\ ,\ ] : E^3 \to E;\ (x,y,z) \mapsto [x,y,z] \tag{2.1}$$

called the *commutator* and a trilinear operation

$$\langle\ ,\ ,\ \rangle : E^3 \to E;\ (x,y,z) \mapsto \langle x,y,z\rangle \tag{2.2}$$

called the *translator*. The commutator is *left alternative*, in the sense that

$$\forall x,z \in E,\ [x,x,z] = 0. \tag{2.3}$$

The translator satisfies the *Jacobi identity*:

$$\forall x,y,z \in E,\ \langle x,y,z\rangle + \langle y,z,x\rangle + \langle z,x,y\rangle = 0. \tag{2.4}$$

Finally, the commutator and translator together satisfy the *comtrans identity*:

$$\forall x,y \in E,\ [x,y,x] = \langle x,y,x\rangle. \tag{2.5}$$

A submodule $J$ of a comtrans algebra $E$ is said to be an *ideal* (notation $J \triangleleft E$) if, for all $j$ in $J$ and $x,y$ in $E$, the elements $[y,x,j]$, $\langle y,x,j\rangle$ and $\langle j,x,y\rangle$ of $E$ lie in $J$. A submodule $J$ of $E$ is an ideal iff it is the kernel of the underlying module homomorphism of a comtrans algebra homomorphism with domain $E$ [7, Prop. 3.1].

A comtrans algebra $E$ is *abelian* if its commutator and translator are identically zero. If the underlying module of a comtrans algebra $E$ is cyclic, then (2.3) and (2.5), along with the trilinearity of the commutator and translator, show that the algebra $E$ is abelian. Note that, for an abelian comtrans algebra $E$, each submodule of $E$ forms an ideal of $E$. At the opposite extreme, a comtrans algebra is said to be *simple* if it is non-abelian, and if it has no proper non-trivial ideals.

Given a comtrans algebra $E$ with commutator $[\ ,\ ,\ ]$ and translator $\langle\ ,\ ,\ \rangle$, a new comtrans algebra $E^{\tau}$, called the *transpose* of $E$, may be formed by equipping the underlying module $E$ with a new commutator

$$[x,y,z]^{\tau} = [z,y,x] + \langle y,z,x\rangle \tag{2.6}$$

and a new translator

$$\langle x,y,z\rangle^{\tau} = -\langle x,z,y\rangle \tag{2.7}$$

for $x,y,z$ in $E$. Note that $E^{\tau\tau} = E$. The algebras $E$ and $E^{\tau}$ are "term equivalent" in the sense of universal algebra, so they have the same (congruences and) ideals [10, p.13]. Moreover, given comtrans algebras $E$ and $F$, a module homomorphism $f : E \to F$ is a comtrans algebra homomorphism $f : E \to F$ iff $f : E^{\tau} \to F^{\tau}$ is a comtrans algebra homomorphism.

## 3. Forms and algebras

A *formed module* $(E,\beta)$ is a unital module $E$ over a commutative ring $R$ with 1, equipped with a bilinear form $\beta : E \times E \to R$. Associated with $(E,\beta)$ is the comtrans algebra $CT(E,\beta)$ having commutator

$$[x,y,z] = y\beta(x,z) - x\beta(y,z) \tag{3.1}$$

and translator

$$\langle x,y,z\rangle = y\beta(z,x) - x\beta(y,z). \tag{3.2}$$

There is also the transposed comtrans algebra $CT(E,\beta)^{\tau}$ with commutator

$$[x,y,z]^{\tau} = z(\beta(x,y) - \beta(y,x)) \tag{3.3}$$

and translator

$$\langle x,y,z\rangle^{\tau} = x\beta(z,y) - z\beta(y,x). \tag{3.4}$$

The *radical* of $(E,\beta)$ is the submodule

$$Rad\,\beta = \{x \in E \mid \forall y \in E,\ \beta(x,y) = \beta(y,x) = 0\}. \tag{3.5}$$

**Proposition 3.1.** *The radical of $(E, \beta)$ is an ideal of $CT(E, \beta)$ and $CT(E, \beta)^{\tau}$.*

*Proof.* Since the algebras $CT(E, \beta)$ and $CT(E, \beta)^{\tau}$ are term equivalent, they have the same ideals [10, p.13]. It thus suffices to prove $Rad\,\beta \triangleleft CT(E, \beta)$. Consider an element $j$ of $Rad\,\beta$. Then for $x, y$ in $E$, one has $[j, x, y] = \langle j, x, y\rangle = -j\beta(x, y) \in Rad\,\beta$ and $\langle y, x, j\rangle = 0$, as required. □

**Corollary 3.2.** *If either $CT(E, \beta)$ or $CT(E, \beta)^{\tau}$ is simple, then $Rad\,\beta = \{0\}$.*

*Proof.* By the term equivalence of $CT(E, \beta)$ and $CT(E, \beta)^{\tau}$, it suffices to consider $CT(E, \beta)$ alone. If $CT(E, \beta)$ were simple with $Rad\,\beta > \{0\}$, Proposition 3.1 would imply $Rad\,\beta = E$, whence $\beta = 0$ and the contradiction that $CT(E, \beta)$ would be abelian. □

If the underlying module $E$ of a comtrans algebra is cyclic, the algebra is necessarily abelian. In this case, it is clear that all bilinear forms $\beta$ on $E$ yield the same comtrans algebra $CT(E, \beta)$. For tighter connections between formed modules $(E, \beta)$ and the comtrans algebras $CT(E, \beta)$ or $CT(E, \beta)^{\tau}$, some restrictions on the possible modules $E$ are required. The following definition serves to impose such restrictions.

**Definition 3.3.** *A formed space $(E, \beta)$ is a formed module whose underlying module $E$ is free of rank more than 1.*

The simplicity of the comtrans algebra $CT(E, \beta)$ of a formed space may be characterized quite sharply.

**Theorem 3.4.** *Let $(E, \beta)$ be a formed space over a commutative ring $R$ with 1. Then the algebras $CT(E, \beta)$ and $CT(E, \beta)^{\tau}$ are simple if and only if $Rad\,\beta = \{0\}$ and $R$ is a field.*

*Proof.* To begin, suppose that $CT(E, \beta)$ is simple. By Corollary 3.2, $Rad\,\beta = \{0\}$. Suppose that $R$ is not a field, so that it has a proper non-zero ideal $I$. Since $E$

is free of positive rank, the submodule $IE$ is proper and non-trivial. One obtains the contradiction $IE \triangleleft CT(E,\beta)$. Consider an element $j = i_1x_1 + \cdots + i_nx_n$ of $IE$ with $i_1,\ldots,i_n \in I$ and $x_1,\ldots,x_n \in E$. Then for $y,z$ in $E$ one has $[j,y,z] = y\beta(j,z) - j\beta(y,z) = i_1y\beta(x_1,z) + \cdots + i_ny\beta(x_n,z) - j\beta(y,z) \in IE$. Similarly $\langle j,y,z\rangle \in IE$ and $\langle z,y,j\rangle \in IE$, so that $IE \triangleleft CT(E,\beta)$.

Conversely, suppose that $R$ is a field and $Rad\,\beta = \{0\}$. Let $j$ be a non-zero element of a non-trivial ideal $J$ of $CT(E,\beta)$. Since $j \notin Rad\,\beta$, there is an element $y$ of $E$ such that $\beta(j,y) \neq 0$ or $\beta(y,j) \neq 0$. Consider an arbitrary element $x$ of $E$. If $\beta(j,y) \neq 0$, one has $x = \beta(j,y)^{-1}([j,x,y] + j\beta(x,y)) \in J$. If $\beta(y,j) \neq 0$, one has $x = \beta(y,j)^{-1}(\langle j,x,y\rangle + j\beta(x,y)) \in J$. Thus $J$ is improper: the only ideals of $CT(E,\beta)$ are $\{0\}$ and $E$. Since $dim_R E > 1$, there is a proper non-trivial subspace $K$ of $E$. If $CT(E,\beta)$ were abelian, then $K$ would be an ideal. Thus $CT(E,\beta)$ is non-abelian. □

The radical of a formed space $(E,\beta)$ is determined by the comtrans algebra $CT(E,\beta)$.

**Proposition 3.5.** *Let $(E,\beta)$ be a formed space, with corresponding comtrans algebra $CT(E,\beta)$. Then*

$$Rad\,\beta = \{x \in E \mid \forall y,\ z \in E,\ \langle z,y,x\rangle = 0\} \tag{3.6}$$

*Proof.* Let $B$ be a basis for the free module $E$. Note $|B| > 1$. For any two distinct elements $b,c$ of $B$, and for $x$ in $E$, the equation $0 = \langle c,b,x\rangle = b\beta(x,c) - c\beta(b,x)$ forces $\beta(b,x) = 0 = \beta(x,c)$. Thus $Rad\,\beta$ contains the right hand side of (3.6). Conversely, $x \in Rad\,\beta \Rightarrow \langle z,y,x\rangle = y\beta(x,z) - z\beta(y,x) = 0$ for $y,z$ in $E$. □

Finally, isomorphism of formed spaces is equivalent to isomorphism of the corresponding comtrans algebras.

**Theorem 3.6.** *Let $(E,\beta)$ and $(E,\gamma)$ be formed spaces on the same underlying module $E$. Then the following three conditions on a module automorphism $f : E \to E$ are equivalent:*

(a) *$f : (E,\beta) \to (E,\gamma)$ is an isomorphism of formed modules;*

(b) *$f : CT(E,\beta) \to CT(E,\gamma)$ is a comtrans algebra isomorphism;*

(c) *$f : CT(E,\beta)^{\tau} \to CT(E,\gamma)^{\tau}$ is a comtrans algebra isomorphism.*

*Proof.* The equivalence of the conditions (b) and (c) follows from the term equivalence of each comtrans algebra with its transpose. If (a) holds, the module automorphism $f$ of $E$ is such that $\forall x, y \in E$, $\gamma(xf, yf) = \beta(x,y)$. Let $[\ ,\ ,\ ]_\beta$ and $[\ ,\ ,\ ]_\gamma$ denote the commutators of $CT(E,\beta)$ and $CT(E,\gamma)$ respectively. Then $\forall x, y, z \in E$, $[xf, yf, zf]_\gamma = yf\gamma(xf, zf) - xf\gamma(yf, zf) = yf\beta(x,z) - xf\beta(y,z) = (y\beta(x,z) - x\beta(y,z))f = [x,y,z]_\beta f$. Similarly, one obtains the equation $\langle xf, yf, zf\rangle_\gamma = \langle x,y,z\rangle_\beta f$ for the respective translators$\langle\ ,\ ,\ \rangle_\beta$ and $\langle\ ,\ ,\ \rangle_\gamma$ of $CT(E,\beta)$ and $CT(E,\gamma)$. Thus (b) holds. Conversely, suppose that (b) holds: there is a comtrans algebra isomorphism $f : CT(E,\beta) \to CT(E,\gamma)$. Let $B$ be a basis for the free module $E$. Note that $|B| > 1$, and that $Bf$ is also a basis for $E$. Given elements $b, c$ of $B$, choose an element $a$ of $B$ distinct from $b$ and an element $\theta$ of the dual module $E^*$ with $af\theta = 1$ and $bf\theta = 0$. Then $[a,b,c]_\beta f = [af, bf, cf]_\gamma \Rightarrow (b\beta(a,c) - a\beta(b,c))f = bf\beta(a,c) - af\beta(b,c) = bf\gamma(af,cf) - af\gamma(bf,cf) \Rightarrow \beta(b,c) = (af\beta(b,c) - bf\beta(a,c))\theta = (af\gamma(bf,cf) - bf\gamma(af,cf))\theta = \gamma(bf,cf)$. Thus $f$ is an isomorphism of formed modules, and (a) holds. □

**Corollary 3.7.** *For a formed space $(E,\beta)$, the automorphism groups of $(E,\beta)$, of $CT(E,\beta)$ and of $CT(E,\beta)^{\tau}$ coincide.*

*Proof.* Take $\gamma = \beta$ in Theorem 3.6 □

## 4. Recognizing form algebras

Given a comtrans algebra $E$, under what conditions is there a bilinear form $\beta$ on $E$ such that $E = CT(E, \beta)$ or $E = CT(E, \beta)^{\tau}$? It transpires that transposed form algebras $CT(E, \beta)^{\tau}$ are slightly easier to recognize, but of course $E = CT(E, \beta)$ if and only if $E^{\tau} = CT(E, \beta)^{\tau}$. The main answer to the problem is given by the following theorem.

**Theorem 4.1.** *Let $E$ be a free module over $R$ of rank more than 2. Then there is a bilinear form $\beta$ on $E$ such that $E = CT(E, \beta)^{\tau}$ if and only if the following two conditions obtain:*

(a) $\forall x, y, z \in E,\ [x, y, z] \in zR$ ;

(b) $\forall x, y, z \in E,\ \langle x, y, z\rangle \in xR + zR$.

*Proof.* By (3.3) and (3.4), conditions (a) and (b) are clearly necessary for transposed form algebras $CT(E, \beta)^{\tau}$. Conversely, suppose that $E$ carries a comtrans algebra structure satisfying conditions (a) and (b). Let $B$ be a basis for $E$; in particular $|B| > 2$. By (a),

$$\exists \delta : B^3 \to R.\ \forall b, c, d \in B,\ [b, c, d] = d\delta(b, c, d). \tag{4.1}$$

For $d \neq d', b, c \in B$, there is a scalar $\lambda$ with $(d - d')\lambda = [b, c, d - d'] = [b, c, d] - [b, c, d'] = d\delta(b, c, d) - d'\delta(b, c, d')$, whence $\delta(b, c, d) = \lambda = \delta(b, c, d')$. Thus (4.1) may be rewritten as:

$$\exists \delta : B^2 \to R.\ \forall b, c, d \in B,\ [b, c, d] = d\delta(b, c). \tag{4.2}$$

By the trilinearity and left alternativity of the commutator, $\delta : B^2 \to R$ extends to a skew- symmetric form $\delta : E^2 \to R$ such that

$$\forall x, y, z \in E,\ [x, y, z] = z\delta(x, y). \tag{4.3}$$

By (b),

$$(4.4) \qquad \exists \beta, \gamma : B^3 \to R.\ \forall b, c, d \in B,\ \langle b, c, d\rangle = b\beta(d, c, b) - d\gamma(c, b, d).$$

For a 3-element subset $\{b, b', d\}$ of $B$ and $c$ in $B$, there are scalars $\lambda$ and $\mu$ with $(b - b')\lambda + d\mu = \langle b - b', c, d\rangle = \langle b, c, d\rangle - \langle b', c, d\rangle = b\beta(d, c, b) - b'\beta(d, c, b') - d(\gamma(c, b, d) - \gamma(c, b'd))$, whence $\beta(d, c, b) = \lambda = \beta(d, c, b')$. For a 3-element subset $\{b, d, d'\}$ of $B$ and $c$ in $B$, there are scalars $\xi$ and $\eta$ with $b\xi - (d - d')\eta = \langle b, c, d - d'\rangle = \langle b, c, d\rangle - \langle b, c, d'\rangle = b(\beta(d, c, b) - \beta(d', c, b)) - d\gamma(c, b, d) + d'\gamma(c, b, d')$, whence $\gamma(c, b, d) = \eta = \gamma(c, b, d')$. Then (4.4) may be partly rewritten as:

$$(4.5) \qquad \exists \beta, \gamma : B^2 \to R.\ \forall b \neq d,\ c \in B,\ \langle b, c, d\rangle = b\beta(d, c) - d\gamma(c, b).$$

Suppose $b \neq c \neq d \in B$. By (4.4) and (4.5), the Jacobi identity gives $0 = \langle b, c, d\rangle + \langle c, d, b\rangle + \langle d, b, c\rangle = b\beta(d, c, b) - d\gamma(c, b, d) + c\beta(b, d) - b\gamma(d, c) + d\beta(c, b) - c\gamma(b, d)$. Equating coefficients of $c$ yields

$$(4.6) \qquad \forall b, d \in B,\ \beta(b, d) = \gamma(b, d).$$

Suppose $b \neq d,\ c \in B$. By (4.3), (4.5) and (4.6), the comtrans identity gives $0 = [b, c, d] + [d, c, b] - \langle b, c, d\rangle - \langle d, c, b\rangle = d\delta(b, c) + b\delta(d, c) - b\beta(d, c) + d\beta(c, b) - d\beta(b, c) + b\beta(c, d)$. Equating coefficients of $d$ yields

$$(4.7) \qquad \forall b, c \in B,\ \delta(b, c) = \beta(b, c) - \beta(c, b).$$

Now by (2.5), (4.2) and (4.7), $\langle b, c, b\rangle = [b, c, b] = b\delta(b, c) = b\beta(b, c) - b\beta(c, b)$. Together with (4.5) and (4.6), this yields

$$(4.8) \qquad \forall b, c, d \in B,\ \langle b, c, d\rangle = b\beta(d, c) - d\beta(c, b).$$

Extend $\beta : B^2 \to R$ to a bilinear form on $E$. By (4.3) and (4.7),

$$(4.9) \qquad \forall x, y, z \in E,\ [x, y, z] = z(\beta(x, y) - \beta(y, x)).$$

By (4.8) and the trilinearity of the translator,

$$\forall x, y, z \in E,\ \langle x, y, z\rangle = x\beta(z, y) - z\beta(y, x). \tag{4.10}$$

Comparing (4.9) with (3.3) and (4.10) with (3.4), the algebra $E$ is identified as $CT(E, \beta)^r$. □

The following example shows the necessity of the rank condition in the statement of Theorem 4.1.

**Example 4.2.** Let $(H, \alpha)$ be a hyperbolic plane [5, Def.II.9.7] over a field, generated by the basis $\{e, f\}$ of isotropic vectors with $\alpha(e, f) = -\alpha(f, e) = 1$. Define

$$[x, y, z] = -z\alpha(y, x) \tag{4.11}$$

and

$$\langle x, y, z\rangle = x\alpha(z, y). \tag{4.12}$$

Then $H$ becomes a comtrans algebra, satisfying conditions (a) and (b) of Theorem 4.1. If $H = CT(H, \beta)^r$ for a bilinear form $\beta$ on $H$, (3.4) and (4.12) would yield

$$\forall x, y, z \in H,\ x\alpha(z, y) = x\beta(z, y) - z\beta(y, x). \tag{4.13}$$

For fixed $x, y$, choose $z$ linearly independent of $x$. Then (4.13) would give $\beta(y, x) = 0$, leading to the contradiction that $H$ would be abelian.

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## GENERAL SUMMARY

This thesis is primarily devoted to studying a class of algebras, the so-called comtrans algebras. It consists of two parts.

In the first part, simple comtrans algebras determined by Lie algebras and by pairs of matrices are characterized. It is also shown that the two classes are separate, except for the vector triple product algebra.

In the second part, the equivalence of the representation theory of a comtrans algebra and the representation theory of an associative universal enveloping algebras is established. Furthermore, the universal enveloping algebra of a comtrans algebra $E$ over a field and the tensor algebra over $(E \wedge E) \oplus (E \otimes E) \oplus (E \otimes E)$ are shown to be isomorphic.

In the third part, comtrans algebras $CT(E, \beta)$ and $CT(E, \beta)^\tau$ from a module $(E, \beta)$ with bilinear form are produced. The automorphism groups of the formed space $(E, \beta)$ and of the comtrans algebras $CT(E, \beta)$ and $CT(E, \beta)^\tau$ coincide. Conditions under which a comtrans algebra is a "form algebra" are given.

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